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Efficient, fair, and strategy-proof allocation of discrete resources under weak preferences and constraints

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Abstract

We study no-transfer allocation of indivisible objects under weak preferences and constraints. A case in point is the allocation of time slots for vaccination among residents. Under the assumption that the constraints constitute a discrete structure called an integral polymatroid, we show that our new mechanism is efficient, respects priorities, and is strategy-proof and polynomial-time computable. The mechanism determines a final allocation through an algorithm that adjusts the so-called rank profile in response to excess demands, which is similar in spirit to auction mechanisms.

Keywords: No-transfer allocation, constraints, weak preference relation, efficiency, strategy-proofness, vaccine allocation

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1 Introduction

Receiving vaccinations have become more common than ever during the COVID-19 pandemic. Accelerating vaccination involves the problem of how to distribute the time slots for vaccination among residents. The distribution is often conducted through a centralized system of the following form:[1] an authority sets up dates on which residents can be vaccinated, each resident submits her possible dates in accordance with her preference, and then, the authority decides who is vaccinated on which date. This allocation problem possesses two key features that are typically precluded from the standard model: (i) weak preferences and (ii) constraints. Regarding (i), it is often the case that an agent is available on several dates, i.e., that she is indifferent between them.[2] Regarding (ii), each date has its quota, i.e., the maximum number of residents who can be vaccinated on that day. Furthermore, the sum of vaccinated people throughout the dates cannot exceed the number of available vaccine doses. Determining vaccination venues and dates further complicates the constraints.

Motivated by this allocation problem, the current paper introduces a new mechanism for object allocation problems without monetary transfers under weak preferences and constraints. The critical elements of our model are (1) agents have single-unit demand, (2) preferences are ordinal, (3) assignments are deterministic rather than probabilistic, and (4) constraints are enforced on feasible allocations. We assume that the set of feasible allocations constitutes a discrete structure called an *integral polymatroid*, a concept in discrete mathematics. As will be detailed later, the class of integral polymatroids permits, as a special case, *hierarchical constraints*, which are widely observed in real problems.

We introduce a new mechanism that works under the above assumption on fea-

[1]A centralized system is adopted in British Columbia (Government of British Columbia, 2022) and Washington state (Washinton State Department of Health, 2022), among others.

[2]In general allocation/matching problems, indifferences in preferences emanate from other sources such as a lack of information. For a detailed account, see the Introduction of Erdil and Ergin (2017).

sible allocations. Our main theorem states that the mechanism satisfies desirable properties: it is efficient, respects priorities, and is strategy-proof. Here, the second property of respecting priorities is a fairness condition; assuming that there is a priority order over agents (which is identical across objects), the property guarantees that no agent envies the outcome of an agent with a lower priority. We also show that the mechanism is polynomial-time computable by using the techniques of discrete convex analysis (Murota, 2003). To the best of our knowledge, this study is the first to deal with weak preferences and constraints simultaneously and develop a mechanism with desirable welfare, incentive, and computational properties.

We formulate our mechanism as an algorithm that takes an allocation problem as an input and outputs an allocation. The key to the algorithm is to define "excess demand" for allocation problems without money and iteratively reduce it by adjusting the so-called rank profile. We argue that the algorithm can be viewed as a no-transfer analogue of the auction mechanisms developed by Demange et al. (1986) and Gul and Stacchetti (2000).

Related literature

To handle indifferences inherent in weak preferences, a common approach is to break ties and then apply a mechanism under strict preferences, most notably the *serial dictatorship mechanism* (Satterthwaite and Sonnenschein, 1981).[3] However, this approach is not appealing in our problem, because the resulting mechanism violates efficiency.[4] Svensson (1994) first overcomes the inefficiency problem by developing a new mechanism for allocation problems under weak preferences. His mechanism satisfies efficiency, fairness, and strategy-proofness. Our contribution is to generalize

[3]Svensson (1999) proves that a mechanism is neutral and group strategy-proof if and only if it is a serial dictatorship mechanism. For a generalization of this result, see Pápai (2000) or Pycia and Ünver (2017).

[4]This fact was previously pointed out by Bogomolnaia et al. (2005); see the first paragraph of Section 4 therein.

Svensson's mechanism by accommodating a wider class of constraints while preserving computational efficiency.

Building upon Svensson's (1994) result, Bogomolnaia et al. (2005) characterize a class of mechanisms that satisfy efficiency, strategy-proofness and other desiderata as a selection from the so-called bi-polar serially dictatorial rule. Jaramillo and Manjunath (2012) consider an exchange market under weak preferences and introduce a generalization of Gale's TTC algorithm (Shapley and Scarf, 1974). Our analysis is distinguished from theirs in that constraints and fairness are taken into account. In a recent study, Krysta et al. (2019) identify the maximum possible number of agents who receive a non-null object under a strategy-proof mechanism. While sharing the same interest in weak preferences, they do not incorporate constraints into the analysis.

The issue of indifferences arises not only in object allocation problems but also in matching problems. The most well-known example is school choice, where school priorities over students invlove ties. Erdil and Ergin (2008) and Abdulkadiroğlu et al. (2009) work on how to find a student-optimal stable matching under weak priorities. While these authors assume that the agents on the "many" side (i.e., schools) have weak priorities, we assume that the agents on the "one" side have weak preferences. Erdil and Ergin (2017) develop a two-sided matching model under weak preferences for both sides and introduce a new algorithm that finds a Pareto-efficient stable matching. The key differences from our result are that our mechanism can handle constraints and satisfies strategy-proofness.

A notable feature of our analysis is to utilize the notion of *matroid* for analyzing constrints. Matroid and its variations have been integrated into the notion of *M-convexity* in discrete convex analysis (Murota, 2003). Prior work has revealed that M-convexity is fundamental for running the DA/TTC algorithms under constraints; see Hafalir et al. (2022) and the literature review therein. An integral polymatroid used in our study is a special case of an *$M^\natural$-convex set*, a variation of an M-convex set. Our novelty is to develop a new method for dealing with weak preferences under discrete convex constraints. As noted earlier, matroidal constraints contain hierarchical constraints as a special case, which have been studied in allocation problems with strict preferences; see Budish et al. (2013) or Kamada and Kojima (2018).

While we consider only deterministic assignments of objects, existing studies have also analyzed random assignments. Hylland and Zeckhauser (1979) introduce the pseudo-market mechanism that achieves an efficient outcome. Their ingenious idea is to assign artificial prices for probability shares and iteratively adjust them until demand and supply are brought into balance.[5] In our mechanism, we identify ranks of objects in a preference order as "prices" and iteratively adjust them.

The remainder is organized as follows. Section 2 introduces our model. Section 3 defines our new mechanism and presents the main theorem about its properties. Section 4 concludes. The proof of the main theorem is relegated to Section 5.

2 Model

Let $N = \{1, \ldots, n\}$ be a set of **agents** and let K be a set of **objects** (more precisely, object *types*). There is a special object, called the **null object**, denoted ϕ; let $\bar{K} := K \cup \{\phi\}$. An **allocation** is a vector $\mu := (\mu_i)_{i \in N}$ that assigns object $\mu_i \in \bar{K}$ to agent i. For an allocation μ, we define $x^\mu \in \mathbb{Z}^K_{\geq 0}$ by

$$x^\mu_k = |\{i \in N : \mu_i = k\}| \text{ for all } k \in K,$$

representing the vector of the number of agents who receive each object (except the null object). There is a set of **feasible vectors** $\mathcal{F} \subseteq \mathbb{Z}^K_{\geq 0}$; we assume that $\mathcal{F}$ is nonempty and bounded. An allocation μ is said to be **feasible** if $x^\mu \in \mathcal{F}$. Let $\mathcal{A}$ denote the **set of feasible allocations**.

[5] A recent study by Gul et al. (2020) substantially generalizes Hylland and Zeckhauser's (1979) result by establishing the existence of Walrasian equilibrium in economies with possibly limited transfers and constraints.

We illustrate feasible vectors in the example of allocating time slots for vaccination. Suppose that there are two dates, $K = \{k, \ell\}$, on which residents can get vaccinated. Up to 100 residents can be accommodated on either day, but there are only 150 vaccine doses in total. Then,

$$\mathcal{F} = \left\{x \in \mathbb{Z}^K_{\geq 0} : 0 \leq x_k \leq 100,\ 0 \leq x_\ell \leq 100,\ 0 \leq x_k + x_\ell \leq 150\right\}. \tag{1}$$

Remark 1. We assume $\mathcal{F} \subseteq \mathbb{Z}^K_{\geq 0}$ rather than $\mathcal{F} \subseteq \mathbb{Z}^{\bar{K}}_{\geq 0}$, thus imposing no restriction on the number of the null object allocated to the agents. The underlying assumption is that the null object is not scarce. ∎

Each agent i has a weak (complete and transitive) **preference relation** over $\bar{K}$, denoted $\succsim_i$; let $\succ_i$ and $\sim_i$ denote the strict and indifference relations induced from $\succsim_i$, respectively. We denote by $\mathcal{R}$ the set of all weak preference relations. Let $\succsim := (\succsim_i)_{i \in N} \in \mathcal{R}^N$ denote the **preference profile** of all agents. For $j \in N$, we use the notation $\succsim_{-j} := (\succsim_i)_{i \in N \setminus \{j\}}$.

Following Svensson (1994) and Pathak et al. (2021), we assume that there is a **baseline priority order** $\trianglerighteq$, which is a weak relation over N; let $\triangleright$ denote the induced strict relation. To quote Pathak et al. (2021): "This priority order captures the ethical values guiding the allocation of the scarce medical resources." In the context of time slot allocation for vaccination, if $j \in N$ is an elderly person or essential personnel and $h \in N$ is a young healthy person, then j is given a higher priority than h, which is represented as $j \triangleright h$.[6] Without loss of generality, we assume that

$$1 \trianglerighteq 2 \trianglerighteq \cdots \trianglerighteq n-1 \trianglerighteq n. \tag{2}$$

Namely, an agent with a smaller index has a weakly higher priority.

A **mechanism** $\varphi : \mathcal{R}^N \to \mathcal{A}$ maps preference profiles to feasible allocations. At $\succsim \in \mathcal{R}^N$, agent i is assigned object $\varphi_i(\succsim)$. We focus on the following three properties:

[6] The order $\trianglerighteq$ could represent other fairness considerations such as needs or waiting time.

- φ is **efficient** if, for any $\succsim \in \mathcal{R}^N$, $\varphi(\succsim)$ is efficient at $\succsim$, i.e., there exists no $\mu \in \mathcal{A}$ such that

$$\left[\mu_i \succsim \varphi_i(\succsim) \text{ for all } i \in N\right] \text{ and } \left[\mu_j \succ \varphi_j(\succsim) \text{ for some } j \in N\right].$$

- φ **respects priorities** if, for any $\succsim \in \mathcal{R}^N$, there exist no $j, h \in N$ with $j \rhd h$ such that

$$\varphi_h(\succsim) \succ_j \varphi_j(\succsim).$$

- φ is **strategy-proof** if, for any $\succsim \in \mathcal{R}^N$, there exists no $j \in N$ and $\succsim'_j \in \mathcal{R}$ such that

$$\varphi_j(\succsim'_j, \succsim_{-j}) \succ_j \varphi_j(\succsim).$$

The first and third properties are standard in the mechanism design literature. The second property was introduced by Svensson (1994) under the name of "weak fairness"; Pathak et al. (2021) and Aziz and Brandl (2021) introduce a related property in the context of medical rationing. It states that an agent j never envies the outcome of another agent h who has a strictly lower priority than j. In the context of time slot allocation, if an elderly person j cannot get vaccinated on any of her possible dates, then a young healthy person h with $j \rhd h$ cannot get vaccinated on any of j's possible dates either.

3 New mechanism

This section consists of five subsections. Section 3.1 introduces an additional assumption on $\mathcal{F}$. Section 3.2 introduces preliminary concepts, which are used to define our new mechanism in Section 3.3. Section 3.4 deals with computational issues.

Section 3.5 presents the main result about the properties of the new mechanism.

We introduce two pieces of notation. For $k \in K$, let $\mathbb{1}^k \in \mathbb{Z}^K_{\geq 0}$ denote the ***k*-th unit vector**, i.e., $\mathbb{1}^k_k = 1$ and $\mathbb{1}^k_\ell = 0$ for all $\ell \neq k$. For $L \subseteq K$ and $x \in \mathbb{Z}^K_{\geq 0}$, let $x(L) := \sum_{k \in L} x_k$.

3.1 Integral polymatroid

We say that $\mathcal{F} \subseteq \mathbb{Z}^K_{\geq 0}$ with $\mathcal{F} \neq \emptyset$ is an **integral polymatroid** (Welsh, 1976) if it satisfies the following two conditions:

(M1) For any $x \in \mathcal{F}$ and $y \in \mathbb{Z}^K_{\geq 0}$ with $y \leq x$, it holds that $y \in \mathcal{F}$.

(M2) For any $x, y \in \mathcal{F}$ with $x(K) < y(K)$, there exists $k \in K$ with $x_k < y_k$ such that $x + \mathbb{1}^k \in \mathcal{F}$.

The former condition implies that we deal with an upper bound constraint. The latter condition is the key property of a matroid, stating that a vector x with a smaller coordinate sum than y can move "one step close" to y while staying inside $\mathcal{F}$. Figure 1 illustrates this move by using an example of an integral polymatroid for $K = \{k, \ell\}$.

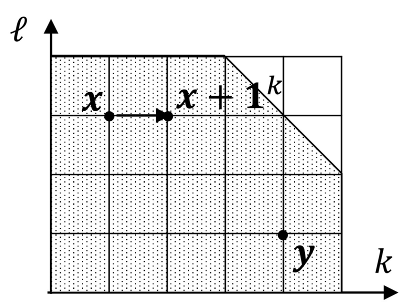

Figure 1. Example of an integral polymatroid for $K = \{k, \ell\}$ (shaded area).

Remark 2. One can verify the following claim: under (M1), if a mechanism φ satisfies efficiency, then it satisfies **individual rationality**, i.e., for any $\succsim \in \mathcal{R}^N$ and any $i \in N$, it holds that $\varphi_i(\succsim) \succsim_i \phi$. ∎

To see a concrete example of an integral polymatroid, we introduce an additional definition. We say that $\mathcal{F}$ is **hierarchical** if:

- there exists a collection of subsets $\mathcal{K} \subseteq 2^K$ with $\mathcal{K} \neq \emptyset$ such that, for any $L, L' \in \mathcal{K}$, either $L \cap L' = \emptyset$ or $L \subseteq L'$ or $L' \subseteq L$ holds; and

- for each $L \in \mathcal{K}$, thete exists $q_L \in \mathbb{Z}_{\geq 0}$ such that

$$\mathcal{F} = \left\{x \in \mathbb{Z}^K_{\geq 0} : x(L) \leq q_L \text{ for all } L \in \mathcal{K}\right\}. \tag{3}$$

One can verify that $\mathcal{F}$ given by (3) is an integral polymatroid. This type of constraints naturally appear in real problems. One such example is provided in (1). Another example is when x vaccine doses are available in January and additional y doses are available in February. In this case, the sum of vaccinated residents in January is no greater than x and the total number of vaccinated residents in January and February is no greater than $x + y$. This case also can be accommodated by hierarchical feasible vectors.

3.2 Rank of objects and requirement function

We introduce key concepts for defining our new mechanism. Fix $\succsim \in \mathcal{R}^N$ throughout this section. For $i \in N$, take an integer $r_i \in \{1, \ldots, |\bar{K}|\}$, which we call a **rank**. We define the set of **top r_i ranked objects** (at $\succsim_i$), denoted $\bar{K}_i(r_i; \succsim_i)$, inductively as follows:

$$\bar{K}_i(1; \succsim_i) = \{k \in \bar{K} : k \succsim_i \ell \text{ for all } \ell \in \bar{K}\},$$
$$\bar{K}_i(r_i; \succsim_i) = \left\{k \in \bar{K} : k \succsim_i \ell \text{ for all } \ell \in \bar{K} \backslash \bar{K}_i(r_i - 1; \succsim_i)\right\} \text{ for all } r_i = 2, \ldots, |\bar{K}|.$$

Intuitively, we gather indifferent objects in batches and then refer to the most preferred batch as the top 1 ranked objects, refer to the most preferred and the second most preferred batches as the top 2 ranked objects, and so on. For example, if $\bar{K} = \{k, \ell, \phi\}$ and 1's preference is $k \sim_1 \ell \succ_1 \phi$, then

$$\bar{K}_1(1; \succsim_1) = \{k, \ell\},\ \bar{K}_1(2; \succsim_1) = \{k, \ell, \phi\},\ \bar{K}_1(3; \succsim_1) = \{k, \ell, \phi\}.$$

We often write $\bar{K}_i(r_i)$ rather than $\bar{K}_i(r_i; \succsim_i)$ when the preference relation is clear from the context. Note that $\bar{K}_i(r_i) \subseteq \bar{K}_i(r'_i)$ whenever $r_i \leq r'_i$.

For $i \in N$, we define i's **requirement function** $\rho_i : 2^K \times \{1, \ldots, |\bar{K}|\} \to \{0, 1\}$ as follows:

$$\rho_i(L, r_i; \succsim_i) = \begin{cases} 1 & \text{if } \bar{K}_i(r_i) \subseteq L, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We often write $\rho_i(L, r_i)$ rather than $\rho_i(L, r_i; \succsim_i)$. In words, $\rho_i(L, r_i) = 1$ means that i requires, or demands, at least one object in L in order to receive an object ranked r_i or higher. Since L is chosen not from $2^{\bar{K}}$ but from 2^K, the following implication holds:

$$\text{for any } L \in 2^K,\ \phi \in \bar{K}_i(r_i) \Longrightarrow \rho_i(L, r_i; \succsim_i) = 0. \tag{5}$$

Take an agent set $\{1, \ldots, m\} \subseteq N$ $(1 \leq m \leq n)$ and a profile $r := (r_i)_{i \in N}$ (called a **rank profile**). We say that **excess demand occurs** at $(\{1, \ldots, m\}, r)$ if there exists $L \in 2^K$ such that

$$\sum_{i=1}^{m} \rho_i(L, r_i) > \max_{x \in \mathcal{F}} x(L). \tag{6}$$

The objects in L are in short supply at r in the sense that we cannot give all the agents in $\{1, \ldots, m\}$ a top r_i ranked object. We say that **excess demand does not occur** at $(\{1, \ldots, m\}, r)$ if there exists no $L \in 2^K$ that satisfies (6).

The requirement function was previously introduced in an auction setting; see

Demange et al. (1986) or Gul and Stacchetti (2000). Their auction algorithms proceed by increasing the prices of the commodities in excess demand. Our novelty is to transfer the technique to a setting without monetary transfers by drawing an analogy between ranks and prices.

We characterize feasible allocations in terms of excess demand.

Proposition 1. *Let $\succsim \in \mathcal{R}^N$ and r be a rank profile. Suppose that $\mathcal{F}$ is an integral polymatroid. Then, the following are equivalent:*

(i) Excess demand does not occur at (N, r).

(ii) There exists $\mu \in \mathcal{A}$ such that $\mu_i \in \bar{K}_i(r_i)$ for all $i \in N$.

Proof. The proof is based on Yokote (2020); see the Appendix. □

This proposition states that the "no excess demand" condition is essential for guaranteeing the existence of a feasible allocation. To make use of this result, our new mechanism iteratively reduces excess demand by adjusting the rank profile.

3.3 Generalized Svensson mechanism

As in the previous section, fix $\succsim \in \mathcal{R}^N$. We are ready to define our new algorithm, the **generalized Svensson mechanism**:

- Step 0: Let $i^0 = 1$ and $r^0 = (1, \ldots, 1)$.
- Step $t \geq 1$:

  (a) If excess demand occurs at $(\{1, \ldots, i^{t-1}\}, r^{t-1})$, then define

$$i^t = i^{t-1},\; r_i^t = \begin{cases} r_i^{t-1} + 1 & \text{if } i = i^{t-1}, \\ r_i^{t-1} & \text{if } i \neq i^{t-1}. \end{cases}$$

  Go to step $t + 1$.

(b) Otherwise, define $i^t = i^{t-1} + 1$ and $r^t = r^{t-1}$.

   * If $i^t \leq n$, then go to step $t+1$.
   * Otherwise, terminate the algorithm and define the outcome as a feasible allocation $\mu \in \mathcal{A}$ such that $\mu_i \in \bar{K}_i(r_i^t)$ for all $i \in N$ (such an allocation always exists by Proposition 1).[7]

The outcome of this mechanism coincides with that of Svensson's (1994) mechanism (up to indifferences) when

$$\mathcal{F} = \Big\{x \in \mathbb{Z}^K_{\geq 0} : x_k \leq 1 \text{ for all } k \in K\Big\}.$$

In Section 3.5, we present an example of how this algorithm proceeds. Here we explain it in words. The algorithm proceeds by iteratively adjusting two variables: the set of agents (controlled by $i^t \in \{1, 2, \ldots, n\}$) and the rank profile (controlled by r^t). We call $\{1, \ldots, i^t\}$ the set of **active agents**; intuitively, these are the agents whose demands are taken into account, where the word "demand" is used in the sense of the requirement function (see the interpretation of the function after (4)). Initialization is $i^0 = 1$ (only agent 1 is active) and $r^0 = (1, \ldots, 1)$ (all the agents demand their first-ranked objects). At each step $t \geq 1$, we check whether excess demand occurs at $(\{1, \ldots, i^{t-1}\}, r^{t-1})$.

- If excess demand occurs (i.e., (a) holds), then we increase the rank of the lowest-priority agent i^{t-1} by 1.

- If excess demand does not occur (i.e., (b) holds), then we expand the set of active agents by increasing i^{t-1} by 1.

The following claim holds:

Claim 1. *Case (a) holds at most* $|\bar{K}| - 1$ *consecutive times.*

[7] If there are multiple feasible allocations, we choose an arbitrary one.

Proof. Suppose that case (b) holds in step $t-1$ and then case (a) holds $|\bar{K}|-1$ consecutive times, i.e., case (a) holds in every step t' with $t \leq t' \leq (t-1)+(|\bar{K}|-1)$. Let $t^* := (t-1)+(|\bar{K}|-1)+1$. We show that case (b) holds in step t^*. Since the i^{t-1}-th coordinate of the rank profile is adjusted $|\bar{K}|-1$ times, we get $r_{i^{t-1}}^{t^*-1} = |\bar{K}|$. Since ϕ is always included in a top $|\bar{K}|$ ranked object, by (5),

$$\rho_{i^{t-1}}(L, r_{i^{t-1}}^{t^*-1}) = 0 \text{ for all } L \in 2^K. \tag{7}$$

Since case (b) holds in step $t-1$, excess demand does not accur at $(\{1,\ldots,i^{t-2}\}, r^{t-2})$ (note that $i^{t-2} = i^{t-1}-1$). By the definition of the algorithm, for any i with $1 \leq i \leq i^{t-2}$, it holds that $r_i^{t-2} = r_i^{t^*-1}$. Therefore, excess demand does not occur at $(\{1,\ldots,i^{t-2}\}, r^{t^*-1})$. Together with (7), excess demand does not occur at $(\{1,\ldots,i^{t-1}\}, r^{t^*-1}) = (\{1,\ldots,i^{t^*-1}\}, r^{t^*-1})$. We conclude that case (b) holds in step t^*. □

Case (b) holds at most n times because there are n agents. Together with the above claim, the number of steps is at most $|\bar{K}-1|\times n$, which is a polynomial function in $|\bar{K}|$ and n. In the next section, we show that the computational time in each step is also bounded by a polynomial function.

3.4 Computational issues

Our mechanism involves two computational problems:

(I) to check whether excess demand occurs or not at every step, and

(II) to find a feasible allocation at the end of the algorithm.

We show that both problems can be solved in time polynomial in the number of agents and objects. To this end, we assume that there is a sufficiently large number $d \in \mathbb{R}$, independent of the number of agents and objects, such that $\max\{\sum_{k\in K} x_k :$

$x \in \mathcal{F}\big\} \leq d$. This means that there is a reasonable bound on the size of feasible vectors, which typically holds true in practice.[8]

We begin with the first problem (I). Since excess demand concerns all subsets of objects $L \in 2^K$, it appears to be a hard problem to check whether excess demand occurs or not. However, this problem turns out to be computationally easy. To see this point, we define the **excess demand function** at $(\{1, \dots, m\}, r)$, denoted $ED : 2^K \to \mathbb{Z}$, as follows:

$$ED(L) = \sum_{i=1}^{m} \rho_i(L, r_i) - \max_{x \in \mathcal{F}} x(L) \text{ for all } L \in 2^K. \tag{8}$$

One easily verifies from (6) that excess demand occurs at $(\{1, \dots, m\}, r)$ if and only if $\max\limits_{L \in 2^K} ED(L) > 0$.

Proposition 2. *$ED(\cdot)$ is a supermodular function, i.e., for any $L, L' \in 2^K$, it holds that $ED(L) + ED(L') \leq ED(L \cup L') + ED(L \cap L')$.*

Proof. It is well-known in the literature on discrete mathematics that, for an integral polymatroid $\mathcal{F}$, $L \mapsto -\max_{x \in \mathcal{F}} x(L)$ is supermodular; see, e.g., Theorem 4.15 of Murota (2003). Since supermodularity is closed under taking the sum of functions, it suffices to prove that, for an arbitrary chosen $i \in N$, $\rho_i(L, r_i)$ is supermodular. Suppose not, i.e., there exist $L, L' \in 2^K$ such that

$$\rho_i(L) + \rho_i(L') > \rho_i(L \cup L') + \rho_i(L \cap L'). \tag{9}$$

If the left-hand side is equal to 1, then $\rho_i(L) = 1$ or $\rho_i(L') = 1$. Together with

$$\rho_i(L) \leq \rho_i(L \cup L') \text{ and } \rho_i(L') \leq \rho_i(L \cup L'), \tag{10}$$

we have $\rho_i(L \cup L') = 1$. Then, the right-hand side of (9) is no less than 1, a contra-

[8]For example, the number of people vaccinated on a single day in one vaccination venue is subject to capacity and operational constraints and does not become arbitrarily large.

diction to the strict inequality. The remaining possibility is that the left-hand side of (9) is equal to 2, which is true only if $\rho_i(L) = \rho_i(L') = 1$. By (10), $\rho_i(L \cup L') = 1$. Furthermore, $\rho_i(L) = \rho_i(L') = 1$ implies $\rho_i(L \cap L') = 1$. Thus, the right-hand side of (9) is equal to 2, a contradiction to the strict inequality. □

It is known that the maximum value of a supermoduolar function can be computed in time polynomial in the number of agents and objects; see, e.g., Section 10.2 of Murota (2003). Thus, given a pair $(\{1, \ldots, m\}, r)$, checking whether $\max_{L \in 2^K} ED(L) > 0$ is true or not can be done in polynomial time.

We can also solve the second problem (II) quickly by formulating it as a so-called *submodular flow problem*. Consider a directed graph $G = (V, A)$ with the set of vertices $V := N \cup \bar{K}$ and the set of arcs $A := N \times \bar{K}$. A function $\xi : A \to \{0, 1\}$ is called a **flow**. For a flow ξ, we define $\partial\xi \in \mathbb{Z}^V$ (called the **boundary of flow $\boldsymbol{\xi}$**) by

$$(\partial\xi)_i = \sum_{k \in \bar{K}} \xi(i, k) \text{ for all } i \in N, \; (\partial\xi)_k = -\sum_{i \in N} \xi(i, k) \text{ for all } k \in \bar{K}.$$

Figure 2 gives an example for $N = \{1, 2, 3\}$ and $\bar{K} = \{k, \phi\}$.

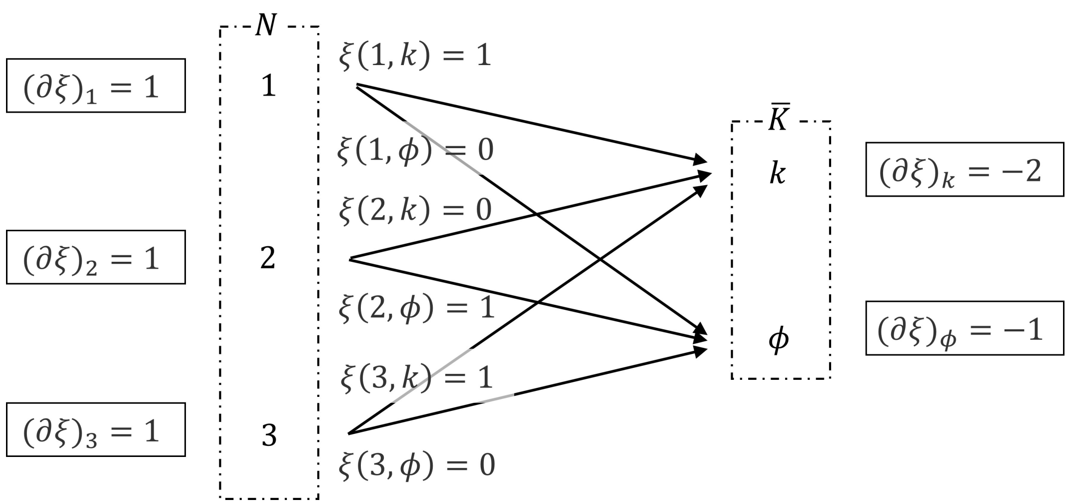

Figure 2. Example of a directed graph, a flow, and its boundary.

We define

$$B_1 = \Big\{x \in \mathbb{Z}^N : x_i = 1 \text{ for all } i \in N\Big\},\ B_2 = \Big\{x \in \mathbb{Z}^{\bar{K}} : (x_k)_{k\in K} \in \mathcal{F}, \sum_{k\in\bar{K}} x_k = n\Big\}.$$

Let $B := B_1 \times (-B_2) \subseteq \mathbb{Z}^V$, where $-B_2 = \{x \in \mathbb{Z}^{\bar{K}} : -x \in B_2\}$. A flow ξ is said to be **feasible** if $\partial\xi \in B$. For a feasible flow ξ and $i \in N$, by the definition of B_1, there exists a unique object $k \in \bar{K}$ such that $\xi(i,k) = 1$; let μ^ξ denote the allocation that gives the unique object to each $i \in N$. Then, for each $k \in K$, $-(\partial\xi)_k$ represents the number of agents who receive object k at allocation μ^ξ. By the definition of B_2, μ^ξ is a feasible allocation. Thus, finding a feasible flow tantamounts to finding a feasible allocation.

If $\mathcal{F}$ is an integral polymatroid, then B forms a discrete structure called an *M-convex set*.[9] Under this assumption, the problem of finding a feasible flow is an instance of a submodular flow problem, for which we can apply an existing algorithm, such as the successive shortest path algorith. The computational time is polynomial in the number of agents and objects; see Section 10.4.2 of Murota (2003) and the literature therein for a detailed account.

3.5 Properties of the new mechanism

Let φ^{GS} denote the generalized Svensson mechanism. We are in a position to state our main theorem.

Theorem 1. *Suppose that $\mathcal{F}$ is an integral polymatroid. Then, φ^{GS} is efficient, respects priorities, and is strategy-proof.*

Proof. See Section 5. □

[9] The definition of M-convexity is given in Section 4 of Murota (2003). An integral polymatroid satisfies a condition called *M$^\natural$-convexity* (see, e.g., Fujishige (2005) or Murota and Shioura (2018)), which implies M-convexity of $-B_2$ (see Section 4.7 of Murota (2003)). Since B_1 consists of a single point, $B = B_1 \times (-B_2)$ is an M-convex set.

It is noteworthy that φ^{GS} does not satisfy *group* strategy-proofness, a stronger notion than strategy-proofness.[10] For $M \subseteq N$, let $\succsim_M := (\succsim_i)_{i \in M}$ and $\succsim_{-M} := (\succsim_i)_{i \in N \setminus M}$.

- φ is **group strategy-proof** if, for any $\succsim \in \mathcal{R}^N$, there exists no $M \subseteq N$ and $\succsim'_M \in \mathcal{R}^M$ such that

$$\varphi_i(\succsim'_M, \succsim_{-M}) \succsim_i \varphi_i(\succsim) \text{ for all } i \in M, \text{ and}$$
$$\varphi_j(\succsim'_M, \succsim_{-M}) \succ_j \varphi_j(\succsim) \text{ for some } j \in M.$$

To see that φ^{GS} violates this condition, let $N = \{1, 2, 3\}$, $\bar{K} = \{k, \ell, \phi\}$, and

$$\mathcal{F} = \left\{ x \in \mathbb{Z}^K_{\geq 0} : x_k \leq 1, x_\ell \leq 1 \right\}.$$

Suppose that the agents have the following true preferences:

- Agent 1: $k \sim \ell \succ \phi$.
- Agent 2: $\ell \succ \phi \succ k$.
- Agent 3: $k \succ \phi \succ \ell$.

Our algorithm proceeds as follows:

- Step 1: $N^0 = \{1\}$, $r^0 = (1, 1, 1)$. Excess demand does not occur. Expand the set of active agents.
- Step 2: $N^1 = \{1, 2\}$, $r^1 = (1, 1, 1)$. Excess demand does not occur. Expand the set of active agents.

[10] This observation is consistent with Ehlers's (2002) theorem stating that there exists no efficient and group strategy-proof mechanism under weak preferences.

- Step 3: $N^2 = \{1, 2, 3\}$, $r^2 = (1, 1, 1)$. Excess demand occurs.[11] Increase r_3^2 from 1 to 2.

- Step 4: $N^3 = \{1, 2, 3\}$, $r^3 = (1, 1, 2)$. Excess demand does not occur. The algorithm terminates with final allocation $\mu_1 = k$, $\mu_2 = \ell$, $\mu_3 = \phi$.

Now, suppose that 1 and 3 collude and submit the following preferences:

- Agent 1: $\ell \succ \phi \succ k$.

- Agent 3: $k \succ \phi \succ \ell$ (same as the true preference).

Then, our algorithm proceeds as follows:

- Step 1: $N^0 = \{1\}$, $r^0 = (1, 1, 1)$. Excess demand does not occur. Expand the set of active agents.

- Step 2: $N^1 = \{1, 2\}$, $r^1 = (1, 1, 1)$. Excess demand occurs.[12] Increase r_2^1 from 1 to 2.

- Step 3: $N^2 = \{1, 2\}$, $r^2 = (1, 2, 1)$. Excess demand does not occur. Expand the set of active agents.

- Step 4: $N^3 = \{1, 2, 3\}$, $r^3 = (1, 2, 1)$. Excess demand does not occur. The algorithm terminates with final allocation $\mu_1 = \ell$, $\mu_2 = \phi$, $\mu_3 = k$.

Compared to the allocation under the true preferences, agent 1 is indifferent and agent 3 becomes strictly better off, showing that group strategy-proofness is violated.

[11] For $L = \{k, \ell\}$, the three agents require one of the two objects (agent 1 requires k or ℓ, agent 2 requires ℓ, and agent 3 requires k), but only two units are available in total (object k has one unit and object ℓ has one unit). Theorefore, excess demand occurs.

[12] For $L = \{\ell\}$, the two agents require object ℓ, but only one unit is available. Therefore, excess demand occurs.

4 Conclusion

In this paper we have developed an efficient, priority-respecting, and strategy-proof mechanism when preferences involve indifferences and constraints are imposed on feasible allocations. The key idea is to reject an agent as being not qualified for object k only if excess demand occurs whenever the agent receives k. Our methodological contribution is to show that ranks of an object in a preference ordering can be identified with "prices" in allocation problems with money. This idea might prove useful in other allocation problems with ordinal preferences.

5 Proof of Theorem 1

Throughout this section, we abbreviate "generalized Svensson mechanism" as GS.

Proof of efficiency: Fix $\succsim \in \mathcal{R}^N$. Suppose for a contradiction that $\varphi^{GS}(\succsim)$ is not efficient. Then, there exists $\mu \in \mathcal{A}$ such that every agent receives a weakly better object than that under $\varphi^{GS}(\succsim)$ and at least one agent receives a strictly better object. For each $i \in N$, we define r_i^* by

$$r_i^* = \min\Big\{r_i \in \{1, \ldots, |\bar{K}|\} : \mu_i \in \bar{K}_i(r_i)\Big\}.$$

Let t be the first step of GS under $\succsim$ at which $r_j^{t-1} = r_j^*$ and $r_j^t > r_j^*$ for some $j \in N$; since there is at least one agent who strictly preferes the object under μ than that under $\varphi^{GS}(\succsim)$, such a step t always exists. Since t is the first step, we have

$$r_i^{t-1} \leq r_i^* \text{ for all } i = 1, \ldots, j-1. \tag{11}$$

By the definition of r^* and the fact that every agent weakly prefers the object under μ than that under $\varphi^{GS}(\succsim)$,

$$r_i^{t-1} \geq r_i^* \text{ for all } i = 1, \dots, j-1. \tag{12}$$

Combining (11) and (12), together with $r_j^{t-1} = r_j^*$, it holds that

$$r_i^{t-1} = r_i^* \text{ for all } i = 1, \dots, j.$$

This implies that, for any $L \in 2^K$ and any $i = 1, \dots, j$,[13]

$$\rho_i(r_i^{t-1}, L; \succsim_i) = 1 \Longrightarrow \mu_i \in L. \tag{13}$$

Then, for any $L \in 2^K$,

$$\max_{x \in \mathcal{F}} x(L) \geq |\{i \in \{1, \dots, j\} : \mu_i \in L\}| \geq \sum_{i=1}^{j} \rho_i(r_i^*, L; \succsim_i),$$

where the first inequality follows from the fact that μ is a feasible allocation and the second inequality follows from (13). We obtain a contradiction to the fact that excess demand occurs at $(\{1, \dots, j\}, r^{t-1})$.

Proof of respecting priorities: Fix $\succsim \in \mathcal{R}^N$. Suppose for a contradiction that there exist $j, h \in N$ with $j \rhd h$ such that

$$\varphi_h^{GS}(\succsim) \succ_j \varphi_j^{GS}(\succsim).$$

By the assumption (2), we have $h > j$. Let $k^* := \varphi_h^{GS}(\succsim)$. By $k^* \succ_j \varphi_j^{GS}(\succsim)$, there exists a step t of GS under $\succsim$ at which $k^* \in \bar{K}_j(r_j^{t-1})$ and excess demand occurs, i.e.,

[13] To see that (13) holds, we consider the contrapositive. If $\mu_i \notin L$, then there exists a top r_i^* ranked object that is not included in L. Since $r_i^{t-1} = r_i^*$, we get $\rho_i(r_i^{t-1}, L; \succsim_i) = 0$.

there exists $L \in 2^K$ such that

$$\sum_{i=1}^{j} \rho_i(r_i^{t-1}, L) > \max_{x \in \mathcal{F}} x(L). \tag{14}$$

Since excess demand does not occur when the agent set is $\{1, \dots, j-1\}$,

$$\sum_{i=1}^{j-1} \rho_i(r_i^{t-1}, L) \leq \max_{x \in \mathcal{F}} x(L). \tag{15}$$

Since the requirement function takes the value of either 0 or 1,

$$\rho_j(r_j^{t-1}, L) \leq 1. \tag{16}$$

Combining (14)-(16), the inequalities of (15) and (16) reduce to equalities. By (16) (holding as equality),

$$\bar{K}_j(r_j^{t-1}) \subseteq L,$$

which together with $k^* \in \bar{K}_j(r_j^{t-1})$ implies $k^* \in L$. Together with (15) (holding as equality),

$$\sum_{i=1}^{j-1} \rho_i(r_i^{t-1}, L) = \max_{x \in \mathcal{F}} x(L).$$

This implies that, at $\varphi^{GS}(\succsim)$, all the objects in L are allocated exhaustively to the agents in $\{1, \dots, j-1\}$. Since $k^* \in L$, we obtain a contradiction to $k^* = \varphi_h^{GS}(\succsim)$ and $h > j$.

Proof of strategy-proofness: Fix a true preference profile $\succsim \in \mathcal{R}^N$. Suppose for a contradiction that an agent $j \in N$ becomes stricly better off by submitting a false preference $\succsim_j' \in \mathcal{R}$, i.e.,

$$\varphi_j^{GS}(\succsim_j', \succsim_{-j}) \succ_j \varphi_j^{GS}(\succsim).$$

Let r' denote the rank profile at the end of GS under $(\succsim_j', \succsim_{-j})$.

The proof goes in parallel with that of φ respecting priorities. Let $k^* := \varphi_j^{GS}(\succsim_j', \succsim_{-j})$. By $k^* \succ_j \varphi_j^{GS}(\succsim)$, there exists a step t of GS under $\succsim$ at which $k^* \in \bar{K}_j(r_j^{t-1}; \succsim_j)$ and excess demand occurs, i.e., there exists $L \in 2^K$ such that

$$\sum_{i=1}^{j} \rho_i(r_i^{t-1}, L; \succsim_i) > \max_{x \in \mathcal{F}} x(L). \tag{17}$$

Since excess demand does not occur when the agent set is $\{1, \ldots, j-1\}$,

$$\sum_{i=1}^{j-1} \rho_i(r_i^{t-1}, L; \succsim_i) \leq \max_{x \in \mathcal{F}} x(L). \tag{18}$$

Since the requirement function takes the value of either 0 or 1,

$$\rho_j(r_j^{t-1}, L; \succsim_j) \leq 1. \tag{19}$$

Combining (17)-(19), the inequalities of (18) and (19) reduce to equalities. By (19) (holding as equality),

$$\bar{K}_j(r_j^{t-1}; \succsim_j) \subseteq L,$$

which together with $k^* \in \bar{K}_j(r_j^{t-1}; \succsim_j)$ implies $k^* \in L$. Since all the agents in $\{1, \ldots, j-1\}$ submit the same preferences between $\succsim$ and $(\succsim_j', \succsim_{-j})$, it holds that $r_i^{t-1} = r_i'$ for all $i = 1, \ldots, j-1$. Together with (18) (holding as equality),

$$\sum_{i=1}^{j-1} \rho_i(r_i', L; \succsim_i) = \max_{x \in \mathcal{F}} x(L).$$

This implies that, at $\varphi^{GS}(\succsim_j', \succsim_{-j})$, all the objects in L are allocated exhaustively to the agents in $\{1, \ldots, j-1\}$. Since $k^* \in L$, we obtain a contradiction to $k^* = \varphi_j^{GS}(\succsim_j'$ $, \succsim_{-j})$. $\square$

Appendix

We prove Proposition 1 by mimicking the proof of Corollary 1 of Yokote (2020). Fix $\succsim \in \mathcal{R}^N$.

Proof of (ii) $\implies$ (i): By (ii), there exists $\mu \in \mathcal{A}$ such that

$$\mu_i \in \bar{K}_i(r_i) \text{ for all } i \in N. \tag{20}$$

Fix an arbitrary $L \in 2^K$. By (20),

$$\rho_i(L, r_i) = 1 \Longrightarrow \mu_i \in L \text{ for all } i \in N. \tag{21}$$

It follows that

$$\sum_{i \in N} \rho_i(L, r_i) \leq x^{\mu}(L) \leq \max_{x \in \mathcal{F}} x(L),$$

where the first inequality follows from (21) and the second inequality follows from $\mu \in \mathcal{A}$. Since L is arbitrarily chosen, we conclude that excess demand does not occur at (N, r).

Proof of (i) $\implies$ (ii): We prove the contrapositive; suppose that

$$\text{there does not exist a feasible allocation } \mu \in \mathcal{A} \text{ such that } \mu_i \in \bar{K}_i(r_i) \text{ for all } i \in N. \tag{22}$$

Let $X_i(r_i) := \{\mathbb{1}^k : k \in \bar{K}(r_i)\}$ for all $i \in N$, with the notation $\mathbb{1}^\phi = \mathbf{0}$. We denote their Minkowski sum by $X := \sum_{i \in N} X_i(r_i)$.[14] Then, (22) is equivalent to

$$X \cap \mathcal{F} = \emptyset. \tag{23}$$

Since $X_i(r_i)$ for $i \in N$ consists of unit vectors, the set satisfies a notion of discrete convexity called *M$^\natural$-convexity* (see Murota (2003)). By Theorem 4.23 of Murota (2003), the Minkowski sum of M$^\natural$-convex sets is also M$^\natural$-convex, which implies that X is M$^\natural$-convex. Furthermore, it is well known in the literature that an integral polymatroid $\mathcal{F}$ is an M$^\natural$-convex set (see, e.g., Murota and Shioura (2018)). Hence, (23) states that two M$^\natural$-convex sets X and $\mathcal{F}$ are disjoint. By applying the so-called *discrete separation theorem* as in the proof of Corollary 1 of Yokote (2020), there exists $L \in 2^K$ such that

$$\min_{x \in X} \mathbb{1}^L \cdot x > \max_{x \in \mathcal{F}} \mathbb{1}^L \cdot x,$$

where $\mathbb{1}^L \in \{0,1\}^K$ denotes the characteristic vector of L, i.e., $\mathbb{1}^L_k = 1$ if $k \in L$ and $\mathbb{1}^L_k = 0$ otherwise. The above strict inequality establishes that (6) holds true, i.e., excess demand occurs at (N, r). $\square$

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[14]For two sets $X_i(r_i)$ and $X_j(r_j)$, their Minkowski sum is defined as $X_i(r_i) + X_j(r_j) = \{x + x' : x \in X_i(r_i), x' \in X_j(r_j)\}$.

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