UTMD Working Paper

The University of Tokyo
Market Design Center

UTMD-035

Efficient Matching under General Constraints

Kenzo Imamura
The University of Tokyo

Yasushi Kawase
The University of Tokyo

November 18, 2022

Efficient matching under general constraints*

Kenzo Imamura and Yasushi Kawase

The University of Tokyo, Tokyo, Japan
{imamurak@e,kawase@mist.i}.u-tokyo.ac.jp

Abstract

We study indivisible goods allocation problems under constraints and provide algorithms to check whether a given matching is Pareto efficient. We first show that the serial dictatorship algorithm can be used to check Pareto efficiency if the constraints are matroid. To prove this, we develop a generalized top trading cycles algorithm. Moreover, we show that the matroid structure is necessary for obtaining all Pareto efficient matchings by the serial dictatorship algorithm. Second, we provide an extension of the serial dictatorship algorithm to check Pareto efficiency under general constraints. As an application of our results to prioritized allocations, we discuss Pareto improving the deferred acceptance algorithm.

1 Introduction

We study indivisible goods allocation problems, including real-life applications such as student placement in public schools and refugee resettlement. These applications are often subject to constraints. A school district requires specific diversity of the student body at each school (*type-specific quotas* and *proportional constraints*). A school needs at least a certain number of students to operate (*minimal quotas*). In refugee resettlement, the central authority needs to consider heterogeneous family sizes and numerous additional requirements such as job training, language class, etc. (*multidimensional constraints*). In a school district, multiple school programs often share one building. There is a limit on the total number of students in these programs in addition to each program's capacity because the building also has a physical capacity (*regional quotas*). Framing the allocation problem as a matching between students and schools, we examine the Pareto efficiency (for students) in a general model of matching with constraints, including all these constraints. Pareto efficiency is

*We are grateful to Keisuke Bando, Fuhito Kojima, M. Bumin Yenmez, and Yu Yokoi for their helpful comments. This work was partially supported by JSPS KAKENHI Grant Number JP20K19739, JST PRESTO Grant Number JPMJPR2122, and Value Exchange Engineering, a joint research project between Mercari, Inc. and the RIISE.

desirable in an allocation problem and has received substantial attention from both theoretical and practical perspectives (see Section 1.2 for details).

In this paper, we study how to check whether a given matching is Pareto efficient or not. Checking Pareto efficiency is one of the fundamental steps in developing a mechanism that satisfies desirable properties. For example, consider the deferred acceptance (DA) mechanism, which may produce a Pareto inefficient matching for students. Thus, we would like to examine whether a current matching produced by the DA is Pareto efficient. Moreover, if the current matching is Pareto inefficient, the subsequent interest would be to Pareto improve the current matching. We provide mechanisms that find such a Pareto improvement.

Checking Pareto efficiency under constraints is non-trivial. In a standard model with capacity constraints, we can check the Pareto efficiency of a given matching by repeatedly applying the following procedure: find a student who is assigned to her best school among the available ones and fix the assignment of the student.[1] If the given matching is Pareto efficient, the process can continue till the end; if the given matching is Pareto inefficient, the process cannot continue because such a student will not exist in some steps. The reason why the procedure works stems from the property that any Pareto efficient matching can be produced by the serial dictatorship (SD) with some order of the students. Here, the order of the students corresponds to the one obtained in the above procedure. This property does not hold under a general constraint, that is, some Pareto efficient matchings cannot be produced by the SD. The following example illustrates this fact.

Example 1. Suppose there are three students i_1, i_2, i_3 and two schools s_1, s_2. The preference $\succ_i$ of student i is as follows:

$$s_2 \succ_{i_1} s_1, \ s_1 \succ_{i_2} s_2, \text{ and } s_2 \succ_{i_3} s_1.$$

The constraint $\mathcal{F}_s$ of school s is as follows:

$$\mathcal{F}_{s_1} = \big\{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_1, i_3\}\big\} \text{ and } \mathcal{F}_{s_2} = \big\{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}\big\}.$$

Here, $\mathcal{F}_{s_2}$ is a capacity constraint, but $\mathcal{F}_{s_1}$ is not. In fact, $\{i_1, i_3\} \in \mathcal{F}_{s_1}$ and $\{i_1, i_2\}, \{i_2, i_3\} \notin \mathcal{F}_{s_1}$. Constraints like $\mathcal{F}_{s_1}$ appear as a budget constraint (e.g., i_2 has a disability and incurs more cost) or an anti-bullying constraint (e.g., i_2 has bullied i_1 and i_3 and cannot be in the same place with them).

Let us consider the matching $\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_1)\}$. Note that μ is feasible and Pareto efficient (which can be verified by examining all possible matchings). However, the SD does not produce μ with any order since no student is assigned to her best school at μ. Thus, we cannot check the Pareto efficiency of μ by the SD.[2]

[1] This procedure can also be interpreted as the top trading cycle (TTC) mechanism, that is, a self-loop is selected in each iteration of the mechanism.

[2] One might think that we can use the top trading cycle (TTC) algorithm to check Pareto

1.1 Our contribution

This paper provides SD-type algorithms to check whether a given matching is Pareto efficient under given constraints. First, we identify which feature of $\mathcal{F}_{s_1}$ in Example 1 prevents the SD from finding the Pareto efficient matching μ: $\mathcal{F}_{s_1}$ is not a *matroid*. Specifically, we show that the SD can find all Pareto efficient matchings if the constraint is a matroid (even when it is distributional). To prove this, we develop a generalized top trading cycles (TTC) algorithm. Moreover, we prove the converse direction: the matroid structure is also a necessary condition to find all Pareto efficient matchings via the SD.

Our second result is an algorithm to check Pareto efficiency under general constraints, including the constraint $\mathcal{F}_{s_1}$ in Example 1 and all constraints listed in the first paragraph of Introduction. Our algorithm, which we call the constrained serial dictatorship (CSD), is based on the SD. Unlike the SD, the CSD is defined on the set of feasible matchings and works under general constraints. In the CSD, a student is assigned to her best school to the extent that the remaining students can be feasibly assigned. The CSD can be used to check the Pareto efficiency of a given matching. To do this, we add an individual rational constraint to the set of feasible matchings in the CSD: every student is weakly better than a given matching. If a matching produced by the CSD is the same as a given matching, then it is Pareto efficient. We also discuss the computational aspects of CSD.

Subsequently, we study a model of indivisible goods allocation with priorities. As applications of our generalized TTC and the CSD, we provide mechanisms that Pareto improve the DA. If the constraint induced by a priority (or a choice function) forms a matroid, then we can apply our generalized TTC by setting the DA matching as an initial matching. This mechanism Pareto improves the DA and is Pareto efficient. In the model with general priorities (path-independent choice functions), we can use the CSD to Pareto improve the DA. Moreover, the CSD is Pareto efficient.

1.2 Related literature

This paper contributes to the literature of matching with constraints. There are two approaches to represent constraints. The first approach considers constraints imposed separately on individual schools. The second approach considers distributional constraints, which are imposed jointly on subsets of schools. This paper considers both those approaches.

The class of matroid constraints, which we first study, subsumes many constraints in the literature. A typical example of individual matroid constraints is the *type-specific quotas*, which is introduced for diversity concerns in school choice [1, 2]. As a distributional matroid constraint, Kamada and Kojima [3]

---

efficiency, but this is not the case. Suppose that μ is the initial matching. While i_1 (or i_3) and i_2 can be better off by trading their assignments, $\{i_2, i_3\}$ (or $\{i_1, i_2\}$) is not feasible for s_1. As a result, the top trading cycles algorithm produces an infeasible matching. Thus, due to the constraint, the TTC algorithm does not work.

study *regional quotas* and introduce a stability concept inspired by medical residency matching in Japan. Kamada and Kojima [4] consider a more general matroidal case where regions are hierarchical. Fleiner [5, 6] shows some results including the existence of a stable matching under the matroid constraints.

General constraints that are not a matroid have also been studied in the literature. Recently, Kamada and Kojima [7] introduce *general upper-bound*. The following constraints studied in recent works as real-life applications belong to general upper-bound: refugee resettlement [8], college admissions with budgets constraints [9], school choice with bullying [10], and daycare allocation [11]. Lower bound constraints [12] and proportionality constraints [13] are examples that do not belong to general-upper bound. A class of generalized-matroid (g-matroid) constraints has a good property as matroid constraints and includes upper and lower bound constraints [14].

The TTC algorithm introduced by Shapley and Scarf [15] can be used to check Pareto efficiency for the unit demand case, and is generalized to some cases [16, 1, 17, 8]. Suzuki et al. [17] generalize the TTC for the case that the distributional constraints are represented as an M-convex set, which is a generalization of a matroid. This algorithm is closely related to ours, but our algorithm is applicable to a more general case than theirs. Their algorithm only works for constraints over the numbers of students in different schools, while our algorithm works for constraints that depend on the identity of the students. Note that, unlike our work, they deal with strategic issues.

Some papers studying stable matching under constraints are closely related to our paper. Kojima et al. [18] show that the generalized DA mechanism satisfies several desirable properties when a constraint is a matroid. Goto et al. [19] deal with general upper-bound constraints and develop a stable and strategy-proof mechanism. Relative to the literature, we focus on Pareto efficiency, not stability (or fairness). As noted above, our algorithms would help in developing a mechanism that satisfies desirable properties; thus, these studies are complementary to each other.

This paper also relates to works that study indivisible goods allocation problems by SD-type algorithms. Fragiadakis et al. [20] modify the SD to accommodate minimum quotas. Our model is more general than theirs, and their algorithm is special case of our CSD. However, Fragiadakis et al. [20] also deal with strategic issues. We also provide a characterization of the set of Pareto efficient matchings under matroid constraints by the SD: a matching is Pareto efficient if and only if it is produced by the SD with some order. Thus, our result is a generalization of the corresponding characterization under capacity constraints. This characterization provides a clear understanding of the structure of the set of Pareto efficient matchings and the foundation for several theoretical studies. The equivalence between the random SD and the core from random endowments serves as an example [21, 22, 23, 24, 25]. The equivalence is based on the characterization.[3] Another example is the study by Manea [26]. Motivated by the characterization, Manea [26] investigates the relationship between Pareto

[3] We need to consider all Pareto efficient matchings to get an equivalent mechanism to the core from random endowment. The idea of the core from random endowment consists of two

efficiency and the SD in a general setting where each agent receives multiple goods.

2 Preliminaries

2.1 Model

A market is a tuple $(I, S, (\succ)_{i\in I}, \mathcal{F})$. I is a finite set of students, and S is a finite set of schools. Each student i has a strict preference $\succ_i$ over $S \cup \{\varnothing\}$, where $\varnothing$ means being unmatched (or an outside option). We write $s \succeq_i s'$ if either $s \succ_i s'$ or $s = s'$ holds. $\mathcal{F}$ is a family of subsets of student-school pairs $I \times S$ that reflects distributional constraints of school side.

Let $E = \{(i, s) \in I \times S : s \succ_i \varnothing\}$ be the set of acceptable student-school pairs. A *matching* μ is a subset of E such that each student appears at most one pair of μ; that is, for any $(i, s), (i', s') \in \mu$, we have $s = s'$ if $i = i'$. We write $\mu(s) = \{i \in I : (i, s) \in \mu\}$ for each $s \in S$. Also, for each $i \in I$, we write $\mu(i)$ to denote the partner of i at μ, that is, $\mu(i) = s$ if $(i, s) \in \mu$ and $\mu(i) = \varnothing$ if $(i, s) \notin \mu$ for all $s \in S$. A matching is called *feasible* if $\mu \in \mathcal{F}$. For notational simplicity, we sometimes add an unmatched pair $(i, \varnothing)$ to a matching, but we ignore such a pair, e.g., for a matching μ, we treat $\mu \cup \{(i, \varnothing)\}$ as μ.

A matching μ *Pareto dominates* μ' if $\mu(i) \succeq_i \mu'(i)$ for all $i \in I$ and $\mu(i) \succ_i \mu'(i)$ for some $i \in I$. Let $\mathcal{M}$ be a set of matchings. A matching $\mu \in \mathcal{M}$ is called *Pareto efficient* with respect to $\mathcal{M}$ if there exists no feasible matching $\mu' \in \mathcal{M}$ that Pareto dominates μ.

For a student $i \in I$ and a subset of schools $S' \subseteq S$, we define $\arg\max_{\succ_i} S'$ to be the school or the outside option that i prefers the most among $S' \cup \{\varnothing\}$. Note that if S' is empty, we have $\arg\max_{\succ_i} S' = \varnothing$.

2.2 Constraints

We say that distributional constraints $\mathcal{F}$ is *individual* if there exists $\mathcal{F}_s \subseteq 2^I$ for $s \in S$ such that $\mathcal{F} = \big\{X \subseteq I \times S : X(s) \in \mathcal{F}_s \ (\forall s \in S)\big\}$, where $X(s) = \{i' : (i', s) \in X\}$. We sometimes write $(\mathcal{F}_s)_{s\in S}$ to represent such an individual $\mathcal{F}$. Also, we call $\mathcal{F}$ as the aggregated constraints of $(\mathcal{F}_s)_{s\in S}$.

An important class of constraints is the matroids. A family of subsets $\mathcal{F}$ is a *matroid* if it satisfies the following three properties: (I1) $\emptyset \in \mathcal{F}$, (I2) if $X \in \mathcal{F}$ and $X' \subseteq X$, then $X' \in \mathcal{F}$, and (I3) if $X, Y \in \mathcal{F}$ and $|X| < |Y|$, there is $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{F}$. We note that an aggregated constraint $\mathcal{F}$ is also a matroid if $\mathcal{F}_s$ is a matroid for every $s \in S$.

A special case of matroid constraints is a *capacity constraint*: a feasibility of a school $\mathcal{F}_s$ is a capacity constraint if there exists a positive integer q such that $\mathcal{F}_s = \{X \subseteq 2^I : |X| \le q\}$. Matroid constraints include many other constraints

steps. First, it randomly chooses a feasible matching. Then, it improves this initial matching by the TTC. Since any feasible matching can be an initial matching, this mechanism assigns positive probabilities to all Pareto efficient matchings. This characterization guarantees that the SD can find all Pareto efficient matchings.

besides a capacity constraint. Let us provide real-life examples of constraints. A school district requires specific diversity of the student body at each school. Abdulkadiroğlu and Sönmez [1] formalize this requirement is to impose type-specific quotas for each school.[4] Each $\mathcal{F}_s$ is a matroid and thus the aggregate constraints $\mathcal{F}$ is also a matroid. Kamada and Kojima [3] study the regional maximum quotas in the context of medical residency matching in Japan. Under the regional maximum quotas, each school belongs to a region, and there is an upper bound on the number of students who can be matched in each region.

Of course, not all constraints of interest are matroids. Kamada and Kojima [7] introduce *general upper-bound*. A constraint $\mathcal{F}$ belongs to general upper-bound if $X' \subseteq X \in \mathcal{F}$ implies $X' \in \mathcal{F}$ (i.e., property (I2)). Thus, a matroid belongs to general upper-bound. A class of constraints that belongs to general upper-bound but not to the matroid is the budget constraints.[5] Another important class of constraints is the g-matroids (which is also called the $M^\natural$-*convex families* [27]). A nonempty family of subsets $\mathcal{F}$ is a *g-matroid* if for any $X, Y \in \mathcal{F}$ and $e \in X \setminus Y$, we have (i) $X \setminus \{e\}, Y \cup \{e\} \in \mathcal{F}$ or (ii) there is $e' \in Y \setminus X$ such that $(X \setminus \{e\}) \cup \{e'\}$ and $(Y \cup \{e\}) \setminus \{e'\}$ are in $\mathcal{F}$. Note that a g-matroid $\mathcal{F}$ is a matroid if $\emptyset \in \mathcal{F}$.

Before completing this section, we present some properties related to matroids that will be used later. For a matroid $\mathcal{F} \subseteq 2^E$ and $X \subseteq E$, the *contraction* of $\mathcal{F}$ by X is defined as $\mathcal{F}/X \coloneqq \{Y \subseteq E \setminus X : Y \cup X \in \mathcal{F}\}$. It is well known that the set family $\mathcal{F}/X$ is also a matroid.

Lemma 1 (see, e.g., [28])**.** Let $(E, \mathcal{F})$ be a matroid with $X \in \mathcal{F}$. Suppose that $Y \subseteq E$, $|Y| = |X|$, and the bipartite graph $G = (X, E \setminus X; \{(x, y) \in X \times (E \setminus X) : (X \setminus \{x\}) \cup \{y\} \in \mathcal{F}\})$ contains a unique perfect matching between $X \setminus Y$ and $Y \setminus X$. Then, $Y \in \mathcal{F}$.

2.3 Serial dictatorship mechanism

Now, we introduce the *serial dictatorship* mechanism (SD) for our setting. The SD mechanism, which is formally described in Algorithm 1, considers the students in a certain order and assigns to each student her best school among available ones. Let Σ be the set of all permutations of the students. For $\sigma \in \Sigma$, we denote SD^σ to be the outcome matching of the SD mechanism.

Since the SD mechanism allocates students greedily, it does not always output a feasible matching in general.[6] To overcome this issue, we define a modified version of the SD, which we call *constrained serial dictatorship* (CSD). Different

[4] Formally, there exist a partition of students $(I_t)_{t \in T}$ with types T, i.e., $I = \bigcup_{t \in T} I_t$ and $I_t \cap I_{t'} = \emptyset$ for any distinct $t, t' \in T$. A feasible constraint $\mathcal{F}_s$ for a school s satisfies the following: there exist a capacity $q \in \mathbb{N}$ and a type-t quota $q_t \in \mathbb{N}$ for every $t \in T$ such that $\mathcal{F}_s = \{X \subseteq I : |X| \leq q \text{ and } |X \cap S_t| \leq q_t\}$.

[5] Formally, there exist weights $(a_i)_{i \in S}$ and a budget b, a feasible constraint $\mathcal{F}_s$ for a school s is represented as $\{X \subseteq I : \sum_{i \in X} a_i \leq b\}$.

[6] For example, let us consider a market $(I, S, (\succ)_{i \in I}, \mathcal{F})$ where $I = \{i_1, i_2\}$, $S = \{s\}$, s is acceptable only for both students, and $\mathcal{F} = \big\{\{(i_1, s), (i_2, s)\}\big\}$. Then, we have $\mathrm{SD}^\sigma = \emptyset$ for any $\sigma \in \Sigma$.

Algorithm 1: SD

input : matching market $(I, S, (\succ)_{i \in I}, \mathcal{F})$, $\sigma \in \Sigma$
output: a matching SD^{σ}

Let $\mu^{(0)} \leftarrow \emptyset$;
for $k \leftarrow 1, 2, \ldots, |I|$ **do**
  Let $r \leftarrow \arg\max_{\succ_{\sigma(k)}} \{s \in S : \mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{F}\}$;
  if $r \in S$ **then** $\mu^{(k)} \leftarrow \mu^{(k-1)} \cup \{(\sigma(k), r)\}$;
return $\mu^{(n)}$;

from the SD, the CSD allocates each student to her most preferred school under the constraint that a feasible matching exists. Fixing a set of feasible matching $\mathcal{M}$, the CSD mechanism is formally described in Algorithm 2. We will set $\mathcal{M}$ not only as the set of school feasible matchings, but also as the set of school feasible matchings that are also individually rational for a given matching.

In social choice literature, a serial dictatorship is defined in a more abstract way and includes the CSD as a special case (see, e.g., [29]). Specifically, by extending a preference of agent to a weak preference over the set of feasible matchings, we can define a "serial dictatorship" in the context of social choice. Algorithm 2 provides an implementation of this abstract rule in our setting. Moreover, the CSD is a natural extension of the SD, which is more familiar in market design literature.

Algorithm 2: constrained serial dictatorship (CSD)

input : $\sigma \in \Sigma$, preference profile $\succ$, feasible matchings $\mathcal{M}$
output: a matching $\mathrm{CSD}^{\sigma,\mathcal{M}}$

Let $\mu^{(0)} \leftarrow \emptyset$;
for $k \leftarrow 1, 2, \ldots, |I|$ **do**
  Let $\mathcal{M}^{(k)} = \big\{\mu \cap (\{\sigma(1), \ldots, \sigma(k)\} \times S) : \mu \in \mathcal{M}\big\}$;
  Let $r \leftarrow \arg\max_{\succ_{\sigma(k)}} \big\{s \in S : \mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{M}^{(k)}\big\}$;
  if $r \in S$ **then** $\mu^{(k)} \leftarrow \mu^{(k-1)} \cup \{(\sigma(k), r)\}$;
return $\mu^{(n)}$;

It is not difficult to see that the SD and the CSD are equivalent (i.e., $\mathrm{SD}^{\sigma} = \mathrm{CSD}^{\sigma,\mathcal{M}}$) if $\mathcal{F}$ belongs to general upper-bound and $\mathcal{M}$ is the set of school feasible matchings.

Theorem 1. If $\mathcal{F}$ belongs to general upper-bound, the outcome matching SD^{σ} is Pareto efficient for any $\sigma \in \Sigma$.

Proof. To obtain a contradiction, suppose that $\mu \coloneqq \mathrm{SD}^{\sigma}$ is Pareto inefficient for some $\sigma \in \Sigma$. Let μ' be a feasible matching that Pareto dominates μ and let k^* be the minimum index such that $\mu'(\sigma(k^*)) \neq \mu(\sigma(k^*))$. By definition, we have $\mu'(\sigma(k)) = \mu(\sigma(k))$ for all $k = 1, 2, \ldots, k^* - 1$ and $\mu'(\sigma(k^*)) \succ_{\sigma(k^*)} \mu(\sigma(k^*))$.

As $\mu' \in \mathcal{F}$ and $\mathcal{F}$ belongs to general upper-bound, the student $\sigma(k^*)$ must be matched with a school that is no worse than $\mu'(\sigma(k^*))$ at line 4 in Algorithm 1, a contradiction. □

3 Condition to check Pareto efficiency by SD: matroid

In this section, we characterize the condition that Pareto efficiency of a given matching can be determined by the serial dictatorship mechanism (Algorithm 1). To be precise, we prove that any Pareto efficient matching can be obtained by the serial dictatorship mechanism if $\mathcal{F}$ is a matroid. Also, we show that a Pareto efficient matching may not be obtained by the serial dictatorship mechanism if $\mathcal{F}$ is not a matroid.

Let us first observe that checking Pareto efficiency of a given matching by TTC is not straightforward even when $\mathcal{F}$ is a matroid.

Example 2. Let $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s_1, s_2\}$. Suppose that i_1 and i_2 prefer s_2 to s_1, while i_3 and i_4 prefer s_1 to s_2. The constraint $\mathcal{F}$ is a (partition) matroid that is defined by the aggregation of

$$\mathcal{F}_{s_1} = \big\{I' \subseteq I : |I' \cap \{i_1, i_3\}| \leq 1 \text{ and } |I' \cap \{i_2, i_4\}| \leq 1\big\},$$
$$\mathcal{F}_{s_2} = \big\{I' \subseteq I : |I' \cap \{i_1, i_4\}| \leq 1 \text{ and } |I' \cap \{i_2, i_3\}| \leq 1\big\}.$$

Consider a matching $\mu = \{(i_1, s_1), (i_2, s_1), (i_3, s_2), (i_4, s_2)\}$. Note that μ is not Pareto efficient because it is Pareto dominated by $\mu^* = \{(i_1, s_2), (i_2, s_2), (i_3, s_1), (i_4, s_1)\}$.

Now, we try to apply a TTC-like mechanism to the market of Example 2. A natural implementation of TTC in our setting is to repeatedly execute the following procedure: (i) each student points to students who are matched to her most preferred school, (ii) identify a cycle, (iii) implement the trade indicated by this cycle, and (iv) remove all the involved students. The cycle obtained by this process at μ is (i_1, i_3), but the matching after the trade by this cycle, i.e., $\{(i_1, s_2), (i_2, s_1), (i_3, s_1), (i_4, s_2)\}$, is not feasible.

The reason why the above mechanism fails is that it does not take into account the feasibility of the schools. To resolve this issue, we construct a trading graph on $I \times (S \cup \{\varnothing\})$, instead of I. Our TTC mechanism is formally defined in Algorithm 3. Throughout the algorithm, $\mu \cup \tilde{\mu}$ represents the current matching, and $\tilde{\mu}$ represents the fixed part. Intuitively, each student i (with her current partner $\mu(i)$) points to her most preferred school q_i among those that can accept her by rejecting some student. Then, each student-school pair (i, q_i) determines the matched pairs $(i', \mu(i'))$ such that $(\mu \setminus \{(i', \mu(i'))\}) \cup \{(i, q_i)\} \in \mathcal{F}$. Note that the graph must contain at least one cycle because each node has out-degree at least one. Implementing the trade indicated by this cycle must yield a matching. For the instance in Example 2, $C = ((i_1, s_1), (i_1, s_2), (i_4, s_2), (i_4, s_1), (i_2, s_1), (i_2, s_2), (i_3, s_2))$ is the unique cycle in the graph constructed in the first round. Implementing the trade indicated by C yields μ^*.

Here, selecting a shortest cycle is crucial to keep feasibility of the school side when $\mathcal{F}$ is a matroid.[7] For example, let us consider a market that is almost the same as the one in Example 2, but the school constraints is a matroid $\mathcal{F}'$ which is defined as the aggregation of

$$\mathcal{F}'_{s_1} = \big\{I' \subseteq I : |I' \cap \{i_3, i_4\}| \leq 1 \text{ and } |I'| \leq 2\big\} \text{ and } \mathcal{F}'_{s_2} = \big\{I' \subseteq I : |I'| \leq 2\big\}.$$

Note that $\mathcal{F}'$ is also a matroid. Then C is also a cycle in the graph constructed in the first round. However, implementing the trade indicated by C yields μ^*, which is infeasible. From a computational perspective, such a shortest cycle can be found in linear time ($O(|U|+|V|+|E|) = O(|I|\cdot|S|)$ time) by the breadth-first search algorithm.

Algorithm 3: Top Trading Cycle (TTC)

input : a market $(I, S, (\succ)_{i\in I}, \mathcal{F})$ and a feasible matching μ
output: a matching TTC^μ

Let $\tilde{\mu} \leftarrow \emptyset$ and $R \leftarrow I$;
while $R \neq \emptyset$ **do**
  Construct a bipartite graph $G = (U, V; E)$ as follows:
  - $U = \{u_i : i \in R\}$ where $u_i = (i, \mu(i))$,
  - $V = \{v_i : i \in R\}$ where $v_i = \big(i, \arg\max_{\succ_i} \{s \in S : \{(i, s)\} \in \mathcal{F}/\tilde{\mu}\}\big)$,
  - $E = \big\{(u_i, v_i) : i \in R\big\} \cup \big\{(v, u) \in V \times U : (\mu \setminus \{u\}) \cup \{v\} \in \mathcal{F}/\tilde{\mu}\big\}$;

  Identify a shortest cycle in G, and let $(u_{\tau(1)}, v_{\tau(1)}, \ldots, u_{\tau(k)}, v_{\tau(k)})$ be the cycle;
  Let $\mu \leftarrow \mu \setminus \{u_{\tau(1)}, \ldots, u_{\tau(k)}\}$ and $\tilde{\mu} \leftarrow \tilde{\mu} \cup (\{v_{\tau(1)}, \ldots, v_{\tau(k)}\}$;
  Remove all the involved students, i.e., $R \leftarrow R \setminus \{\tau(1), \ldots, \tau(k)\}$;
return $\tilde{\mu}$;

In what follows, we prove that our TTC mechanism always outputs a Pareto efficient matching TTC^μ that Pareto dominates the given matching μ if $\mathcal{F}$ is a matroid. Moreover, we show that SD^σ is equal to TTC^μ when we set σ as the order that the students are removed in the TTC mechanism. As a consequence, we conclude that any Pareto efficient matching can be obtained by the SD if $\mathcal{F}$ is a matroid.

We first prove that TTC^μ Pareto dominates μ.

Lemma 2. If $\mathcal{F}$ is a matroid, then TTC^μ is a feasible matching that Pareto dominates the given matching μ.

[7] For constraints more general than matroids, selecting a shortest cycle does not guarantee the feasibility of the school side.

Proof. We show that, at the beginning of any iteration of the while-loop in Algorithm 3, $\mu \cup \tilde{\mu}$ is feasible matching that Pareto dominates μ. In the first round, the statement clearly holds from the definition of μ and $\tilde{\mu}$. In each iteration, $\mu \cup \tilde{\mu}$ remains contained in $\mathcal{F}$ by Lemma 1. In addition, the partner of a student i at $\mu \cup \tilde{\mu}$ does not get worse because $\{(i, \mu(i))\}$ must be in $\mathcal{F}/\tilde{\mu}$ if $\mu(i) \in S$ by $\mu \cup \tilde{\mu} \in \mathcal{F}$. □

Next, we show that TTC^{μ} can be represented as SD^{σ} where σ is the order that the students are removed in the TTC mechanism.

Lemma 3. If $\mathcal{F}$ is a matroid, $\mathrm{SD}^{\sigma} = \mathrm{TTC}^{\mu}$ when we set σ as the order in which the students are removed within the TTC mechanism.

Proof. We prove the lemma by induction on the while-loop iterations. At the beginning, $\tilde{\mu} = \emptyset$, which is consistent the outcome of the SD when no student is assigned. In each iteration, student $\tau(j)$ ($j \in \{1, \dots, k\}$) is assigned to $\mathrm{TTC}^{\mu}(\tau(j)) = \arg\max_{\succ_{\tau(j)}} \left\{s \in S : \{(\tau(j), s)\} \in \mathcal{F}/\hat{\mu}\right\}$. Meanwhile, the SD mechanism assigns $\tau(j)$ to $\mathrm{SD}^{\sigma}(\tau(j)) = \arg\max_{\succ_{\tau(j)}} \left\{s \in S : \{(\tau(j), s)\} \in \mathcal{F}/(\hat{\mu} \cup (\{v_{\tau(1)}, \dots, v_{\tau(j-1)}\})\right\}$. Here, as TTC^{μ} is feasible by Lemma 2, we have $\{(\tau(j), \mathrm{TTC}^{\mu}(\tau(j)))\} \in \mathcal{F}/(\hat{\mu} \cup (\{v_{\tau(1)}, \dots, v_{\tau(j-1)}\})\}$. Hence, we obtain $\mathrm{TTC}^{\mu}(\tau(j)) = \mathrm{SD}^{\sigma}(\tau(j))$. Therefore, $\mathrm{SD}^{\sigma} = \mathrm{TTC}^{\mu}$. □

Theorem 2. If $\mathcal{F}$ is a matroid, we have

$$\{\mathrm{SD}^{\sigma} : \sigma \in \Sigma\} = \{\mu : \mu \text{ is a Pareto efficient matching}\}.$$

Proof. By Theorem 1, it is sufficiently to prove that any Pareto efficient matching μ can be represented as SD^{σ} for some $\sigma \in \Sigma$. Fix such an efficient matching μ. By Lemma 2, TTC^{μ} must be equal to μ. Thus, by taking σ as shown in Lemma 3, we can conclude that $\mathrm{SD}^{\sigma} = \mathrm{TTC}^{\mu} = \mu$. □

Conversely, we show that the matroid structure is a necessary condition to represent every Pareto efficient matching as an outcome of the SD mechanism. The class of matroid constraints subsumes many practical cases, but there are some constraints that are not matroid, such as school admissions with budget constraints. A natural question is whether the conclusion of Theorem 2 holds without the assumption of matroid. The following result shows that the answer to this question is negative.

Theorem 3. Fix a set of students I, a set of schools S with $|S| \geq 2$, and a school $s \in S$ and its constraint $\mathcal{F}_s$. Suppose $\mathcal{F}_s$ is not a matroid. Then there exist a preference profile $(\succ)_{i \in I}$ and a capacity constraint profile $\mathcal{F}_{-s}$ such that

$$\{\mathrm{SD}^{\sigma} : \sigma \in \Sigma\} \neq \{\mu : \mu \text{ is a Pareto efficient matching}\}. \tag{1}$$

Proof. First, suppose that $\mathcal{F}_s$ violates property (I1). Then we have

$$\{\mathrm{SD}^{\sigma} : \sigma \in \Sigma\} = \{\emptyset\} \neq \emptyset = \{\mu : \mu \text{ is a Pareto efficient matching}\}$$

when every school is unacceptable for all students.

Next, suppose that $\mathcal{F}_s$ violates property (I2), i.e., there exist $X, Y \subseteq I$ such that $X \subseteq Y$, $Y \in \mathcal{F}_s$, and $X \notin \mathcal{F}_s$. Consider a preference profile such that each student $i \in Y$ only accepts s and each student $i \notin Y$ accepts no school. Then, there is a unique Pareto efficient matching $\{(i, s) : i \in Y\}$. On the other hand, for σ such that $\sigma^{-1}(x) < \sigma^{-1}(y)$ for any $x \in X$ and $y \in Y \setminus X$, some student in X must be rejected by s in SD^σ. Hence, $\mathrm{SD}^\sigma \neq \{(i, s) : i \in Y\}$ for such σ, and (1) holds.

Finally, suppose that $\mathcal{F}_s$ violates only property (I3), i.e., there exist $X, Y \in \mathcal{F}_s$ such that $|X| < |Y|$ and $X \cup \{i\} \notin \mathcal{F}_s$ for all $i \in Y \setminus X$. Let s' be a school that is different from s. We will choose a capacity constraint $\mathcal{F}_{s'}$, a preference profile $(\succ_i)_{i \in I}$, and a matching μ such that

- μ is Pareto efficient;
- $\mathrm{SD}^\sigma \neq \mu$ for any $\sigma \in \Sigma$.

By $Y \in \mathcal{F}_s$ and (I2), there exists $Z \subseteq X \cup Y$ such that $Z \in \mathcal{F}_s$ and $|Z| = |X| + 1$. Choose $Z \in \arg\max_{Z \subseteq X \cup Y} \{|Z \cap X| : Z \in \mathcal{F}_s,\ |Z| = |X| + 1\}$. Note that $X \not\subseteq Z$ by the assumption that $X \cup \{i\} \notin \mathcal{F}_s$ for all $i \in Y \setminus X$. Let $\mathcal{F}_{s'}$ be a capacity constraint with capacity $|X \setminus Z|$. Construct a matching μ and a preference profile $(\succ_i)_{i \in I}$ as follows:

- for $i \in X \setminus Z$, $s \succ_i s' \succ_i \varnothing \succ_i t$ for any school $t \in S \setminus \{s, s'\}$ and $\mu(i) = s'$,
- for $i \in Z \cap X$, $s \succ_i \varnothing \succ_i t$ for any school $t \in S \setminus \{s\}$ and $\mu(i) = s$,
- for $i \in Z \setminus X$, $s' \succ_i s \succ_i \varnothing \succ_i t$ for any school $t \in S \setminus \{s, s'\}$ and $\mu(i) = s$,
- for $i \in I \setminus (X \cup Z)$, $\varnothing \succ_i t$ for any school $t \in S$ and $\mu(i) = \varnothing$.

We show that μ is Pareto efficient by contradiction. Suppose that μ' Pareto dominates μ. As μ' is a feasible matching, we have $\mu'(s) \subseteq X \cup Z$ and $\mu'(s) \in \mathcal{F}_s$. Also, as μ' Pareto dominates μ, we have $\mu'(i) = s$ for all $i \in Z \cap X$ and $\mu'(i) = s$ for some $i \in X \setminus Z$. Hence, we have $|\mu'(s) \cap X| > |Z \cap X|$. This implies $|\mu'(s)| < |Z|$ by the maximality of Z. Thus, we obtain $|\mu'(s')| = |X \cup Z| - |\mu'(s)| > |X \cup Z| - |Z| = |X \setminus Z|$, which contradicts the feasibility of μ'. What is left is to show that $\mathrm{SD}^\sigma \neq \mu$ for any $\sigma \in \Sigma$. If $\sigma^{-1}(i) < \sigma^{-1}(i')$ for some $i \in Z \setminus X$ and $i' \in X \setminus Z$, we have $\mathrm{SD}^\sigma(i) = s'$ but $\mu(i) = s$. Otherwise (i.e., $\sigma^{-1}(i) > \sigma^{-1}(i')$ for any $i \in Z \setminus X$ and $i' \in X \setminus Z$), we have $\mathrm{SD}^\sigma(i') = s$ for all $i' \in X \setminus Z$ by $X \in \mathcal{F}_s$ and (I2), but $\mu(i') = s'$. Hence, $\mathrm{SD}^\sigma \neq \mu$ for any $\sigma \in \Sigma$. □

This result shows that Theorem 2 cannot be generalized further in the sense that the class of matroids is a "maximal domain." Specifically, a matroid is the most permissive restriction on constraints imposed on individual schools which guarantees that the SD finds all Pareto efficient matchings. Note that, if $\mathcal{F}_s$ violates only property (I3), we can replace (1) in Theorem 3 with

$$\{\mathrm{SD}^\sigma : \sigma \in \Sigma\} \subsetneq \{\mu : \mu \text{ is a Pareto efficient matching}\}$$

by Theorem 1.

4 Checking Pareto efficiency

In this section, we show that Pareto efficiency of a given matching can be checked by the CSD mechanism (Algorithm 2) for general constraints. Specifically, we apply the CSD with $\mathcal{M}$ as the set of school feasible matchings that are also individually rational for the given matching. We prove that the outcome of the CSD is a Pareto efficient matching that weakly Pareto dominates the given matching. As a consequence, we can conclude that Pareto efficiency of a given matching can be checked by the CSD.

Let us first observe that if a matching is Pareto efficient under an individual rational constraint, then the matching is also Pareto efficient without the constraint.

Lemma 4. For a set of matchings $\mathcal{M}$ and a matching $\mu \in \mathcal{M}$, let $\mathcal{M}' = \{\mu' \in \mathcal{M} : \mu'(i) \succeq_i \mu(i)\ (\forall i \in I)\}$. If μ is Pareto efficient within $\mathcal{M}'$, then it is also Pareto efficient within $\mathcal{M}$.

Proof. We prove the contraposition. Let $\mu^* \in \mathcal{M}'$ be a matching that is Pareto inefficient within $\mathcal{M}$. Then, there exists a matching $\mu' \in \mathcal{M}$ that is a Pareto improvement of μ^*. Here, μ' must be in $\mathcal{M}'$ because $\mu'(i) \succeq_i \mu^*(i) \succeq_i \mu(i)$ for every $i \in I$. Hence, μ^* is also Pareto inefficient within $\mathcal{M}'$. □

Theorem 4. Let $\mathcal{M}$ be the set of school feasible matchings that are also individually rational for a given matching μ. Suppose that $\mathcal{M} \neq \emptyset$. Then, for any $\sigma \in \Sigma$, the matching $\mathrm{CSD}^{\sigma,\mathcal{M}}$ is Pareto efficient and weakly Pareto dominate μ.

Proof. By Lemma 4, it is sufficient to prove that $\mathrm{CSD}^{\sigma,\mathcal{M}}$ is Pareto efficient within $\mathcal{M}$. To obtain a contradiction, suppose that $\mu \coloneqq \mathrm{SD}^{\sigma}$ is Pareto inefficient for some $\sigma \in \Sigma$. Let $\mu' \in \mathcal{M}$ Pareto dominates μ and let k^* be the minimum index such that $\mu'(\sigma(k^*)) \neq \mu(\sigma(k^*))$. By definition, we have $\mu'(\sigma(k)) = \mu(\sigma(k))$ for all $k = 1, 2, \ldots, k^* - 1$ and $\mu'(\sigma(k^*)) \succ_{\sigma(k^*)} \mu(\sigma(k^*))$. As $\mu' \in \mathcal{M}$, the student $\sigma(k^*)$ must be matched with a school that is no worse than $\mu'(\sigma(k^*))$ at line 5 in Algorithm 2, a contradiction. □

Finally, let us discuss computational issues. We observe that Algorithm 2 can be implemented to run in polynomial time if $\mathcal{F}$ is a g-matroid (provided a membership oracle and an initial independent set). To do this, it is sufficient to show that the condition "$\mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{M}^{(k)}$" at line 4 can be checked in polynomial time. We reduce the checking problem to the g-matroid intersection problem defined as follows: given two g-matroids $\mathcal{F}_1$ and $\mathcal{F}_2$, determining whether $\mathcal{F}_1 \cap \mathcal{F}_2$ is nonempty. The g-matroid intersection problem can be solved in polynomial time [30, 31]. Define

$$\mathcal{F}' = \left\{ X \subseteq I \times S : \begin{array}{l} |X \cap (\{i\} \times S)| \leq 1\ (\forall i \in I), \\ (i, s) \notin X \text{ if } \mu(i) \succ_i s\ (\forall i \in I, \forall s \in S), \\ \mu^{(k-1)} \cup \{(\sigma(k), s)\} \subseteq X \end{array} \right\},$$

which is a g-matroid representing the set of individually rational matchings that contains $\mu^{(k-1)} \cup \{(\sigma(k), s)\}$. Hence, the condition "$\mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{M}^{(k)}$" holds if and only if $\mathcal{F} \cap \mathcal{F}'$ is nonempty.

Unfortunately, it is coNP-hard to distinguish whether a given matching is Pareto efficient or not even when $\mathcal{F}_s$ is a budget constraint for each $s \in S$. To observe this, we reduce the *subset sum problem*. The problem is known to be NP-complete [32] and defined as below: given positive integers $a_1, a_2, \ldots, a_\ell$ and a_0 $(\leq \sum_{j=1}^{\ell} a_j)$, deciding whether there exists a subset $J \subseteq \{1, 2, \ldots, \ell\}$ such that $\sum_{j \in J} a_j = a_0$. Given an instance of the subset sum problem, we construct a matching market where $I = \{i_0, i_1, \ldots, i_\ell\}$, $S = \{s_1, s_2\}$, $s_1 \succ_{i_0} s_2$, $s_2 \succ_{i_j} s_1$ $(j = 1, 2, \ldots, \ell)$, and

$$\mathcal{F}_{s_1} = \left\{ I' \subseteq I : \textstyle\sum_{i_j \in I'} a_j \leq \sum_{j=1}^{\ell} a_j \right\} \text{ and } \mathcal{F}_{s_2} = \left\{ I' \subseteq I : \textstyle\sum_{i_j \in I'} a_j \leq a_0 \right\}.$$

Consider a matching $\mu = \{(i_0, s_2), (i_1, s_1), \ldots, (i_\ell, s_1)\}$. If the subset sum instance a yes-instance, i.e., there exists $J \subseteq \{1, 2, \ldots, \ell\}$ such that $\sum_{j \in J} a_j = a_0$, then $\{(i, s_1) : i \in I \setminus J\} \cup \{(i, s_2) : i \in J\}$ Pareto dominates μ. On the other hand, if a matching μ' Pareto dominates μ, then the subset sum instance must be a yes-instance since $\sum_{(i_j, s_2) \in \mu'} a_j = a_0$ and $(i_0, s_2) \notin \mu'$. Hence, μ is not Pareto efficient if and only if the subset sum instance a yes-instance.

Theorem 5. Checking Pareto efficiency of a given matching is coNP-hard even if $\mathcal{F}$ is individual budget constraints.

Although Pareto efficiency is computationally hard to check, the CSD can be implemented to run practically fast in most cases by using an integer programming solver such as Gurobi or CPLEX. For example, for individual budgets constraints, the condition $\arg\max_{\succ_{\sigma(k)}} \left\{ s \in S : \mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{M}^{(k)} \right\} \in S \cup \{\varnothing\}$ at line 4 can be done computed by the following integer programming:

$$\begin{array}{lll}
\max & \sum_{s \in S} \sum_{s' \in S:\, s' \succeq s} x_{\sigma(k), s'} & \\
\text{s.t.} & \sum_{s \in S} x_{i,s} \leq 1 & \forall i \in I, \\
& \sum_{i \in I} a_{i,s} x_{i,s} \leq b_s & \forall s \in S, \\
& x_{i,s} = 1 & \forall (i,s) \in \mu^{(k-1)}, \\
& x_{i,s} = 0 & \forall (i,s) \in I \times S \text{ with } \mu(i) \succ_i s, \\
& x_{i,s} \in \{0, 1\} & \forall (i,s) \in I \times S.
\end{array}$$

Here, $x_{i,s}$ means that i is matched with s or not.

5 Application to prioritized allocations

In this section, we study a model of indivisible goods allocation with priorities. As applications of our generalized TTC and the CSD, we provide mechanisms that Pareto improve the DA.

5.1 Model

Suppose that each school s is endowed with a priority, which is represented by a choice function over sets of students. Let $\mathrm{Ch}_s\colon 2^I \to 2^I$ be the choice function of $s \in S$, where $\mathrm{Ch}_s(I') \subseteq I'$ for all $I' \subseteq I$. The aggregate choice function $\mathrm{Ch}\colon 2^{I\times S} \to 2^{I\times S}$ for $(\mathrm{Ch}_s)_{s\in S}$ is defined as

$$\mathrm{Ch}(X) = \big\{(i,s) : s \in S,\ i \in \mathrm{Ch}_s(X(s))\big\} \quad \text{for } X \subseteq I \times S.$$

A matching μ is called *individually rational* if $\mathrm{Ch}(\mu) = \mu$. Note that this condition can be seen as the feasibility of the school side. A matching μ is *stable* if it is individually rational and there exists no $(i,s) \in I \times S$ such that $s \succ_i \mu(i)$ and $i \in \mathrm{Ch}_s(\mu(s) \cup \{i\})$. The stability leads a fairness notion in a model of indivisible goods allocation with priorities [33].

We introduce conditions that restrict the priorities. A choice function Ch satisfies:

- *substitutability* if for every $x \in X \subseteq Y$, $x \in \mathrm{Ch}(Y)$ implies $x \in \mathrm{Ch}(X)$,
- *path-independence* if for every X and Y, $\mathrm{Ch}(X \cup Y) = \mathrm{Ch}(\mathrm{Ch}(X) \cup \mathrm{Ch}(Y))$, and
- *the law of aggregate demand (LAD)* if for every $X \subseteq Y$, $|\mathrm{Ch}(X)| \leq |\mathrm{Ch}(Y)|$.

Path-independence leads to substitutability, and substitutability and LAD together lead to path-independence. For each property, the aggregate choice function satisfies the property if all the choice functions of the schools satisfy the property.

The *responsive* choice functions are the most standard ones that satisfies substitutability and LAD. For school s, let q_s denote the capacity and $\succ_s$ denote a priority order of the students. The responsive choice function Ch_s with respect to q_s and $\succ_s$ is defined as

$$\mathrm{Ch}_s(I') = \begin{cases} I' & \text{if } |I'| \leq q_s, \\ \{i \in I' : i \succeq_s i^*\} & \text{otherwise,} \end{cases}$$

where $i^* \in I'$ is the student such that $|\{i \in I' : i \succeq_s i^*\}| = q_s$.

When every choice function satisfies path-independence, a stable matching exists, and the DA, which is formally defined in Algorithm 4, finds a stable matching [34, 35]. Moreover, it finds the *student-optimal* stable matching, i.e., the stable matching that Pareto dominates any other stable matchings. Assuming both substitutability and LAD, we obtain some more useful properties.[8]

[8] The DA is strategy-proof; the rural hospital theorem holds; the set of stable matchings is a lattice [36].

Algorithm 4: Deferred Acceptance (DA)

input: a market $(I, S, (\succ)_{i\in I})$ and an aggregate choice function Ch

Let $R \leftarrow I \times S$;

while ***true*** **do**

  $Y \leftarrow \{(i, \arg\max_{\succ_i}\{s : (i, s) \in R\}) : i \in I\}$;

  $Z \leftarrow \mathrm{Ch}(Y)$ and $R \leftarrow R \setminus (Y \setminus Z)$;

  if $Y = Z$ **then return** Y;

5.2 Pareto improvement from the DA

While the DA is one of the most important mechanisms in practice and theory, it may produce a Pareto inefficient matching for students under a path-independent choice function. We use our generalized TTC and the CSD to Pareto improve the DA for students. In what follows, we consider Pareto efficiency only with respect to students.

The DA is weakly Pareto efficient if the choice function satisfies path-independence and LAD [37].[9] However, it is still Pareto inefficient. The size of the matching is unchanged by any Pareto improvement from the DA. The efficiency-adjusted deferred acceptance (EADA) mechanism Pareto improves the DA and is Pareto efficient [38, 39].[10] If a constraint induced by a choice function forms a matroid (e.g., matroidal choice functions [42]), then there is another way of improvement based on our result: apply our generalized TTC by setting the DA matching as an initial matching.[11] This mechanism Pareto improves the DA and is Pareto efficient. Pápai [43] and Alcalde and Romero-Medina [44] study this mechanism in the model with responsive choice functions.

Without LAD, inefficiency of the DA would be worse. The DA is not even weakly Pareto efficient. Moreover, we could obtain a lager matching that Pareto dominates the one produced by the DA. The following example illustrates this fact.

Example 3. Let $I = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2\}$. Suppose that i_1 and i_3 prefer s_2 to s_1, while i_2 prefer s_1 to s_2. School s_1 has a quota $q_{s_1} = 1$ and a weak priority order $i_1 \succ_{s_1} i_2 \sim_{s_1} i_3$. Also, school s_2 has a quota $q_{s_2} = 1$ and a strict priority order $i_2 \succ_{s_2} i_3 \succ_{s_2} i_1$. The choice function Ch_{s_2} is responsive. The choice function Ch_{s_1} is defined as $\mathrm{Ch}_{s_1}(I')$ is I' if $|I'| \leq q_{s_1}$ and $\{i \in I' : i \succeq_{s_1} i^*\}$ otherwise, where i^* is the highest priority student such

[9] A matching μ *strictly Pareto dominates* μ' if $\mu(i) \succ_i \mu'(i)$ for all $i \in I$. A matching $\mu \in \mathcal{M}$ is called *weakly Pareto efficient* with respect to $\mathcal{M}$ if there exists no feasible matching $\mu' \in \mathcal{M}$ that strictly Pareto dominates μ.

[10] Kesten [38] introduce the EADA in the model with responsive choice functions. Bando [40] and Tang and Yu [41] propose equivalent mechanisms to the EADA. Ehlers and Morrill [39] generalize EADA to the settings where choice functions satisfy path-independence and LAD.

[11] In general, a constraint induced by path-independence and LAD is not necessarily a matroid.

that $|\{i \in I' : i \succeq_{s_1} i^*\}| \geq q_{s_1}$.[12] Note that Ch_{s_1} satisfies path-independence, but not LAD.

The matching of the DA is

$$\mu = \{(i_1, s_1), (i_2, s_2)\}.$$

On the other hand,

$$\nu = \{(i_1, s_2), (i_2, s_1), (i_3, s_1)\}$$

strictly Pareto dominates μ and is Pareto efficient.

The size of matching is a crucial criterion in some applications such as refugee resettlement [47]. Therefore, Pareto improving from the DA would be beneficial in this setting. Even without LAD, the EADA is well-defined and Pareto improves the DA. However, it is Pareto inefficient. Intuitively, this is because Pareto efficiency of the EADA relies on weak Pareto efficiency of the DA. To be more specific, the EADA identifies an "irrelevant agent" at the matching produced by the DA (e.g., i_3 in Example 3). Using weak Pareto efficiency, we observe that the irrelevant student cannot be better off for any Pareto improvement from the DA. The fact enables us to fix the irrelevant student with the school assigned under the DA. This does not hold without LAD: as illustrated in Example 3, while ν Pareto dominates the matching produced by the DA, the irrelevant student i_3 can be better off.

By Theorem 4, we can use the CSD to Pareto improve the DA by setting the matching produced by the DA as an initial matching. Moreover, the CSD is Pareto efficient. There are various ways to do that. Any matching Pareto improving the DA can be an initial matching. For example, we can also set the matching produced by the EADA as an initial matching. In addition, any ordering of the students works. For example, an ordering can be defined by utilizing the EADA. Specifically, the ordering of the students can be obtained by iteratively processing the following: identify an irrelevant student for the outcome of the DA, insert her to the head of the list, and remove her from the market.

6 Conclusion

We studied methods to check Pareto efficiency in indivisible goods allocation problems under general constraints. We started with the observation that the standard approach by the SD does not work in general. Our first main result characterizes the constraints that the SD can use to check whether a given matching is Pareto efficient. These are matroid constraints. Our result also generalizes the characterization of Pareto efficient matchings by the SD in the standard model with capacity constraints. Our second main result provides an algorithm to check the Pareto efficiency of a given matching under general

[12] This choice function is studied in the college admissions problem with weak priorities [45, 46].

constraints. Finally, as an application of our generalized TTC and the CSD, we study how to Pareto improve the DA in a model with priorities.

A study on the incentive properties of a mechanism under constraints can be considered in the future. Let us fix an order σ of the students. Then, the CSD$^{\sigma,\mathcal{M}}$ is strategy-proof if reporting preferences cannot affect the set of feasible matchings $\mathcal{M}$. However, the CSD is not strategy-proof under general constraints. For example, the CSD would not be strategy-proof by adding the individual rationality constraint to $\mathcal{M}$. Fragiadakis et al. [20] assume no outside option and show that the CSD is strategy-proof in the model with minimal quotas. Thus, a natural question is which class of constraints guarantees the CSD is strategy-proof. More generally, it is important to design a Pareto efficient and strategy-proof mechanism under general constraints. We leave such investigations for future research.

References

[1] A. Abdulkadiroğlu, T. Sönmez, School choice: A mechanism design approach, American economic review 93 (2003) 729–747.

[2] A. Abdulkadiroğlu, College admissions with affirmative action, International Journal of Game Theory 33 (2005) 535–549.

[3] Y. Kamada, F. Kojima, Efficient matching under distributional constraints: Theory and applications, American Economic Review 105 (2015) 67–99.

[4] Y. Kamada, F. Kojima, Stability and strategy-proofness for matching with constraints: A necessary and sufficient condition, Theoretical Economics 13 (2018) 761–793.

[5] T. Fleiner, A matroid generalization of the stable matching polytope, in: International Conference on Integer Programming and Combinatorial Optimization, Springer, 2001, pp. 105–114.

[6] T. Fleiner, A fixed-point approach to stable matchings and some applications, Mathematics of Operations Research 28 (2003) 103–126. doi:10.1287/moor.28.1.103.14256.

[7] Y. Kamada, F. Kojima, Fair matching under constraints: Theory and applications, Rev. Econ. Stud (2019).

[8] D. Delacrétaz, S. D. Kominers, A. Teytelboym, Refugee resettlement, University of Oxford Department of Economics Working Paper (2016).

[9] A. Abizada, Stability and incentives for college admissions with budget constraints, Theoretical Economics 11 (2016) 735–756.

[10] Y. Kasuya, Anti-bullying school choice mechanism design, Technical Report, working paper, 2016.

[11] Y. Okumura, School choice with general constraints: a market design approach for the nursery school waiting list problem in Japan, The Japanese Economic Review 70 (2019) 497–516.

[12] L. Ehlers, I. E. Hafalir, M. B. Yenmez, M. A. Yildirim, School choice with controlled choice constraints: Hard bounds versus soft bounds, Journal of Economic theory 153 (2014) 648–683.

[13] T. Nguyen, R. Vohra, Stable matching with proportionality constraints, Operations Research 67 (2019) 1503–1519.

[14] Y. Yokoi, A generalized polymatroid approach to stable matchings with lower quotas, Mathematics of Operations Research 42 (2017) 238–255.

[15] L. Shapley, H. Scarf, On cores and indivisibility, Journal of mathematical economics 1 (1974) 23–37.

[16] A. Abdulkadiroğlu, T. Sönmez, House allocation with existing tenants, Journal of Economic Theory 88 (1999) 233–260.

[17] T. Suzuki, A. Tamura, M. Yokoo, Efficient allocation mechanism with endowments and distributional constraints, in: Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems, 2018, pp. 50–58.

[18] F. Kojima, A. Tamura, M. Yokoo, Designing matching mechanisms under constraints: An approach from discrete convex analysis, Journal of Economic Theory 176 (2018) 803–833.

[19] M. Goto, F. Kojima, R. Kurata, A. Tamura, M. Yokoo, Designing matching mechanisms under general distributional constraints, American Economic Journal: Microeconomics 9 (2017) 226–62.

[20] D. Fragiadakis, A. Iwasaki, P. Troyan, S. Ueda, M. Yokoo, Strategyproof matching with minimum quotas, ACM Transactions on Economics and Computation (TEAC) 4 (2016) 1–40.

[21] A. Abdulkadiroğlu, T. Sönmez, Random serial dictatorship and the core from random endowments in house allocation problems, Econometrica 66 (1998) 689–701.

[22] T. Sönmez, M. U. Ünver, House allocation with existing tenants: an equivalence, Games and Economic Behavior 52 (2005) 153–185.

[23] P. A. Pathak, J. Sethuraman, Lotteries in student assignment: An equivalence result, Theoretical Economics 6 (2011) 1–17.

[24] G. Carroll, A general equivalence theorem for allocation of indivisible objects, Journal of Mathematical Economics 51 (2014) 163–177.

[25] S. Bade, Random serial dictatorship: the one and only, Mathematics of Operations Research 45 (2020) 353–368.

[26] M. Manea, Serial dictatorship and pareto optimality, Games and Economic Behavior 61 (2007) 316–330.

[27] K. Murota, Discrete Convex Analysis, SIAM, Philadelphia, 2003.

[28] J. Lee, Matroid Intersection, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2004, p. 84–106. doi:10.1017/CBO9780511616655.006.

[29] H. Eraslan, A. McLennan, Strategic candidacy for multivalued voting procedures, Journal of Economic Theory 117 (2004) 29–54.

[30] K. Murota, Valuated matroid intersection I: Optimality criteria, SIAM Journal on Discrete Mathematics 9 (1996) 545–561.

[31] K. Murota, Valuated matroid intersection II: Algorithms, SIAM Journal on Discrete Mathematics 9 (1996) 562–576.

[32] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman New York, 1979.

[33] M. Balinski, T. Sönmez, A tale of two mechanisms: student placement, Journal of Economic theory 84 (1999) 73–94.

[34] A. E. Roth, Stability and polarization of interests in job matching, Econometrica: Journal of the Econometric Society (1984) 47–57.

[35] O. Aygün, T. Sönmez, Matching with contracts: Comment, American Economic Review 103 (2013) 2050–51.

[36] J. W. Hatfield, P. R. Milgrom, Matching with contracts, American Economic Review 95 (2005) 913–935.

[37] F. Kojima, The law of aggregate demand and welfare in the two-sided matching market, Economics Letters 99 (2008) 581–584.

[38] O. Kesten, School choice with consent, The Quarterly Journal of Economics 125 (2010) 1297–1348.

[39] L. Ehlers, T. Morrill, (Il) legal assignments in school choice, The Review of Economic Studies 87 (2020) 1837–1875.

[40] K. Bando, On the existence of a strictly strong Nash equilibrium under the student-optimal deferred acceptance algorithm, Games and Economic Behavior 87 (2014) 269–287.

[41] Q. Tang, J. Yu, A new perspective on Kesten's school choice with consent idea, Journal of Economic Theory 154 (2014) 543–561.

[42] Y. Yokoi, Matroidal choice functions, SIAM Journal on Discrete Mathematics 33 (2019) 1712–1724.

[43] S. Pápai, Matching with minimal priority rights, Technical Report, Concordia University, 2013.

[44] J. Alcalde, A. Romero-Medina, Strategy-proof fair school placement, Available at SSRN 1743082 (2015).

[45] T. Fleiner, Z. Jankó, Choice function-based two-sided markets: stability, lattice property, path independence and algorithms, Algorithms 7 (2014) 32–59.

[46] P. Biró, S. Kiselgof, College admissions with stable score-limits, Central European Journal of Operations Research 23 (2015) 727–741.

[47] T. Andersson, L. Ehlers, Assigning refugees to landlords in sweden: Efficient, stable, and maximum matchings, The Scandinavian Journal of Economics 122 (2020) 937–965.