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# Combining Boston Mechanism with Deferred Acceptance algorithm

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**Abstract**

We study the matching mechanism in a two-stage game that mixes two well-known matching mechanisms, Boston Mechanism(BM) and the Deferred Acceptance algorithm(DA). First, we show that if all organizations have the same preferences for agents they accept, the subgame perfect equilibrium outcome of the two-stage game is agent-optimal stable matching. We then show that at least one of the subgame perfect equilibria of the two-stage game is an agent-optimal stable matching if the condition of Ergin acyclicity is satisfied. Using one of the conditions of Ergin acyclicity, we also show that DA outcome becomes weakly preferable for all agents to the two-stage game outcome.

**Keywords:** two-stage game, DA algorithm, Subgame Perfect Equilibrium, acyclicity

**JEL Classification:** C78,D47,D61,I20

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# 1 Introduction

## 1.1 Overview

Various algorithms have been adopted as methods of allocation. Boston mechanism (BM) and deferred acceptance (DA) algorithm (Gale and Shapley, 1962), in particular, are very popular methods of matching. Those systems have been analyzed in many aspects individually. However, there has not been much discussion on games that combine BM and DA.

In this paper, we discuss a mechanism of allocation composed of BM and DA. Specifically, we consider a game in which agents play BM in the first stage, and only agents who receive nothing in the first stage play DA in the second stage. In this two-stage game, the number of organizations' seats that are allocated to agents in the first stage and the number of organizations' seats that are allocated to agents over the whole game are decided before the first stage begins. After the first stage, agents play the second stage, and the capacity of each organization in the second stage is its capacity over the whole game minus the number of agents it accepted in the first stage.

This mechanism is important for real-life applications. First, the mechanism is thought to be an inter-form between BM and DA. BM has a straightforward structure, and this makes it one of the most widely used matching methods all over the world. On the other hand, as has generally been pointed out, BM does have disadvantages in terms of strategyproofness and stability. Roth (1991) studies the market for residency in the United Kingdom and shows that stability is important to the continued existence of the mechanism, and so it is natural for matching organizers using BM to attempt to adopt a new mechanism to overcome the disadvantages. DA is a prime candidate to replace BM in such cases. However, a sudden change of allocation method comes at a cost, so organizers might try to change the mechanism gradually, and then a matching mechanism that combines BM and DA can be executed during the changeover process. Second, there may be cases in which acceptors deviate from the organized mechanism and try to secure applicants before the procedure begins. As is well known, the outcome derived from DA is acceptors-pessimal stable matching (Gale and Shapley, 1962). Therefore, if a mechanism director who attempts to allocate something by DA does not have enough control over the game, organizations may stray from organized DA and try to adopt agents in other ways. In such cases, BM is simple and one of the plausible alternatives. The model is similar to the game discussed in this paper.

This allocation procedure is actually used in practice. For example, at the University of Tokyo in Japan, this method of allocation is used to decide which department each student enters. At the University of Tokyo, all second-year students decide on what their special fields of study will be in the following years. Previously, the University of Tokyo implemented only BM. A two-stage matching mechanism such as BM followed by DA has been in use since 2017. Under this process, each department first prioritizes all students according to their test scores. Each department then has a different preference profile over students because different departments focus on different subjects. Students also have different preference profiles over departments. In the first stage, each student applies to one department or does not apply to any department. Before the start of the process, the capacity of each sector in the first stage is determined and announced. This capacity is less than or equal to the whole capacity of each department. This way of allocation is BM. In the second stage, only students who have yet to decide which department to enter in the first stage play DA. Then the capacity of each department in the second stage is the capacity of each department over the whole game minus the number filled in the first stage. This paper analyzes the two-stage game in the example just given.

We compare the outcome derived from the subgame perfect equilibrium (SPE) of the two-stage game with the outcome derived from DA in order to reveal the properties of the two-stage game. The reason why DA is adopted for comparison is that the outcome of DA is agent-optimal and stable. In this paper, we think about a two-stage game in which students apply to schools. In this game, there are some equilibria. Then, we find the following interesting properties of the two-stage game. First, if all schools' preference is common, the outcome of the SPEs of the two-stage game is always the same as the outcome of DA. Next, we use the concept of Ergin acyclicity (Ergin, 2002) to clarify the characteristics of the two-stage game. When Ergin acyclicity holds, if DA is executed, the linkage structure of agents' rejection does not become circular and the outcome satisfies Pareto efficiency. Then, if Ergin acyclicity is satisfied, we show that at least one of the outcomes of the SPEs of the two-stage game is the same as the outcome of DA. Using one of the conditions of Ergin acyclicity, we also indicate what the sufficient condition is for ensuring that the outcome of DA is weakly preferable to the two-stage game's SPE outcomes for every student. Furthermore, we show that if students are forced to apply to some schools in the first stage, all of the above characteristics do not hold.

## 1.2 Related literature

BM and DA have mainly been analyzed as independent from each other, but there is not much research on two-stage games that combine both. However, the literature on BM and DA provides us with many beneficial suggestions. Abdulkadiroglu and Sönmez (2003) bring the perspective of many-to-one matching to the school choice problem. They show that agents could have incentives to misrepresent their preferences in BM and the outcomes are not always Pareto efficient. On the other hand, Ergin and Sönmez (2006) show that the set of Nash equilibrium outcomes coincides with the set of stable matchings in BM. This means that, thinking about the SPE of the two-stage game, the matching in the first stage is stable. While the literature is not extensive, there are still some papers about allocation methods composed of two different mechanisms. Westkamp (2013) analyzes the German university admissions system (Boston mechanism followed by organization-proposing DA) and shows that the Nash equilibrium of that game is equal to the stable matching. Moreover, Dur and Kesten (2019) analyze not only the sequential game, like the German university admissions system, but also the two-stage game and show that every stable matching of the two-stage game is also a Nash equilibrium. These references give us some clues for thinking about the two-stage game. We extend the solution concept to the SPE.

Ergin (2002) introduces the concept of Ergin acyclicity. Ergin indicates that if Ergin acyclicity holds, the outcome of DA is consistent. He then shows that if a priority structure of organizations is acyclical, the matching of the rest of the agents and organizations' seats generated by DA after the removal of some agents and organizations' seats who are matched with each other in DA is equal to the matching in the case of no such exogenous exclusions. We, on the other hand, analyze the case of endogenously deciding which agents and organizations' seats to exclude and study the combination of agents and organizations' seats to eliminate.

# 2 Model

## 2.1 Matching model

Let $I$ be a finite set of students and $S$ be a finite set of schools. Each student $i \in I$ has a strict rational preference $\succ_i$ over $S \cup \{\emptyset\}$. Similarly, each school $s \in S$ has a strict rational preference $\succ_s$ over $I \cup \{\emptyset\}$. Here, $\emptyset$ denotes no matching: for example, $\emptyset \succ_i s$ means that student $i$ prefers not to go to any school than to go to the school $s$. Student $i$ is ***acceptable*** to school $s$ if $i \succ_s \emptyset$ and vice versa. We denote the preference of a student or a school as below.

$$\succ_i : s_1, s_2$$

This means that $s_1 \succ_i s_2 \succ_i \emptyset$ and $\emptyset \succ_i s$ for all $s \in S \backslash \{s_1, s_2\}$. Hence, only $s_1$ and $s_2$ are acceptable to $i$.

For $s \in S$, we denote $Q_s \in \mathbb{N}$ as the ***capacity*** of school $s$. No school can accept students beyond its capacity.

A matching $\mu$ is a mapping from $I \cup S$ to $S \cup 2^I \cup \{\emptyset\}$ and satisfies:

1. For all $i \in I, \mu(i) \in S \cup \{\emptyset\}$.
2. For all $s \in S, \mu(s) \in 2^I$.
3. For all $i \in I$ and $s \in S, \mu(i) = s$ if and only if $i \in \mu(s)$.

A matching $\mu$ satisfies ***individual rationality*** if $\mu(i) \succsim_i \emptyset$ for all $i \in I$ and $j \succsim_s \emptyset$ for all $(j, s) \in I \times S$ which satisfies $j \in \mu(s)$. A matching $\mu$ is ***blocked*** by $(i, s) \in I \times S$ if $s \succ_i \mu(i)$ and either (1) $|\mu(s)| < Q_s$ and $i \succ_s \emptyset$, or (2) $|\mu(s)| = Q_s$ and $i \succ_s j$ for some $j \in \mu(s)$ holds. A matching $\mu$ is ***stable*** if it is individually rational and there is no $(i, s)$ which blocks $\mu$.

A matching $\mu$ ***Pareto dominates for students*** another matching $\nu$ if $\mu(i) \succsim_i \nu(i)$ for all $i \in I$ and $\mu(j) \succ_i \nu(j)$ for some $j \in I$.

## 2.2 Two-stage game

Here, we consider the two-stage games in which matching is done by BM in the first stage and by DA in the second stage. Suppose that only students are strategic. In each stage, students act simultaneously, and they have complete information about the capacity of each school and the preferences of all students and schools. This game can be considered a mixture of short BM and DA.

In the first stage, matching is done according to BM. Specifically, the first stage of the game proceeds as follows. Every school $s \in S$ has capacity $q_s \in \mathbb{Z}$ which satisfies $0 \leq q_s \leq Q_s$. Each student can apply to at most one school. Then, each school $s$ accepts students according to its preference $\succ_s$ until either there are no acceptable students who applied to the school left or its capacity $q_s$ is filled. Note that for $i \in I$ and $s \in S$, if $\emptyset \succ_s i$, then school $s$ will not accept, and will instead reject, student $i$. We have supposed that schools are not strategic players, so they follow their true preference. The acceptance in this stage is permanent, not tentative. Students who are rejected or choose not to apply to any schools move on to the second stage.

In the second stage, DA is conducted. Students know the result of the first stage. In this stage, students report their orders of preference. Since DA is a strategy-proof mechanism, in this paper we only consider cases where students honestly report their preferences. This algorithm consists of the following steps.

*Step 1*: Students who are unmatched in the first stage apply to their most preferred school. Let $t_s \in \mathbb{Z}$ be the number of students accepted by a school $s$ in the first stage. Each school $s$ accepts students in the order of its preference until its capacity $Q_s - t_s > 0$ is filled or there remain no acceptable students.

*Step k* ($k = 2, 3, \cdots$): Students who are rejected in step $k-1$ apply to the most preferred school to which they have not yet applied. Students rejected by every school acceptable to them quit and become unmatched. Each school considers all the students who are already accepted and who applied in this step. It accepts students following its order of preference until its capacity $Q_s - t_s$ is filled or there remain no acceptable students. Schools reject students who are not accepted in this step. If there is no rejection, then this algorithm stops. If not, it moves on to Step $k+1$.

Since students report their true order of preference in the second stage, the strategy set $A_i$ of student $i$ is defined as follows:

$$A_i = \emptyset \cup \{s \in S | q_s > 0\}.$$

The strategy of student $i$ is denoted $a_i \in A_i$. If $a_i = \emptyset$, it refers to a strategy where $i$ does not apply to any school in the first stage and declares an honest preference in the second stage. If $a_i \in \{s \in S | q_s > 0\}$, it means that $i$ applies to school $a_i$ in the first stage and reports an honest preference in the second stage. Let $A = \prod_{i \in I} A_i$ be the set of strategy profiles.

If student $i$ and school $s$ are matched finally under the strategy profile $a \in A$, we write $\mu(i; a) = s$ or $\mu(i) = s$ when the strategy profile is clear. The payoff function $u_i : A \to \mathbb{R}$ is the ordinal utility of student $i$ and satisfies $u_i(a) \geq u_i(a')$ if and only if $\mu(i; a) \succsim_i \mu(i; a')$. The two-stage game is a tuple $\langle I, A, (u_i)_{i\in I}\rangle$. In the following, let $\mu$ be the matching of the SPE outcome in the two-stage game unless otherwise noted. The matching generated only by DA with the same preference orders $(\succ_i)_{i\in I}$ and $(\succ_s)_{s\in S}$ is denoted by $\mu^{DA}$.

# 3 Results

## 3.1 Condition under which the two-stage game and DA produce the same outcome

In this section, we describe the sufficient condition under which all SPE outcomes of two-stage matching are equal to student-optimal stable matching. Let $\mu$, an arbitrary matching, be the SPE outcome of a two-stage game.

**Theorem 1.** *If all schools have the same preferences $\succ_s$, $\mu = \mu^{DA}$.*

The proof is outlined as follows. Consider the following matching system. First, students are prioritized in order of the preferences of each school. Next, in order of priority, students select the school they most want to go to from the available schools. Let $\mu^*$ be this matching. One can then prove that this matching is identical to the matching in the SPE outcome in the two-stage game, and also to DA matching.

The reason that this is identical to the SPE outcome in the two-stage game is as follows. In the SPE of the two-stage game, each student is matched to a school that is equally favorable to or more favorable than $\mu^*$, because each student is in a state where changing strategies does not allow for a better school. More precisely, if there are students matched to schools worse than $\mu^*$, at least one of them will be accepted if they apply in the first stage to the same school as the school matched in $\mu^*$. Furthermore, since $\mu^*$ is Pareto efficient for students, if any student matches to a school better than $\mu^*$ as a result of the two-stage game, at least one student will be matched to a school worse than $\mu^*$. Therefore, by a similar argument, there is no student who matches a school better than $\mu^*$ as an SPE outcome in a two-stage game.

The outcome also coincides with DA matching for the following reasons. DA algorithm produces a stable matching, so there is no pair that blocks $\mu^{DA}$. If the results of DA matching and $\mu^*$ do not coincide, then there are students whose preferred school is taken by a student with a lower priority in DA matching. Therefore, there exists a pair that blocks $\mu^{DA}$, contradicting stability.

Although the theorem makes the strong assumption that all schools have the same preferences, it is also a realistic assumption in the following respects. Consider the situation in which the priority over students is decided by the results of centralized exams. In this case, the preferences of each school can be regarded as being the same. Theorem 1 states that in such a realistic case, the SPE outcome of the two-stage game coincides with DA matching.

## 3.2 Ergin acyclicity

The assumption that each school has the same preference profile over students is strong. In the following section, we consider the concept of Ergin acyclicity, which was introduced by Ergin (2002), to extend the discussion to the case where each school has a heterogeneous preference profile. Ergin defines Ergin acyclicity as follows.
Let $U_s(i) = \{j \in I | j \succ_s i\}$. A ***cycle*** is constituted of distinct $s_1, s_2 \in S$ and $i_1, i_2, i_3 \in I$ such that the following are satisfied:

(C) Cycle condition: $i_1 \succ_{s_1} \succ i_2 \succ_{s_1} i_3 \succ_{s_2} i_1$.

(S) Scarcity condition: There exist (possibly empty) disjoint sets of agents $I_{s_1}, I_{s_2} \subset I \backslash \{i_1, i_2, i_3\}$ such that $I_{s_1} \subset U_{s_1}(i_2), I_{s_2} \subset U_{s_2}(i_1), |I_{s_1}| = Q_{s_1} - 1$, and $|I_{s_2}| = Q_{s_2} - 1$.

A priority structure satisfies ***Ergin acyclicity*** if it has no cycles.
Ergin shows the relationship between Ergin acyclicity and a concept called consistency. Here, ***consistency***, as defined by Ergin, is the property that the matches formed by the complementary set of agents and organizations are coincident before and after the exclusion of agents' and seats of organizations' sets from the universal set of agents and organizations' seats. Ergin then shows that the following are equivalent:

(i): The result of DA is Pareto efficient for applicants.
(ii): The result of DA is group strategyproof.
(iii): The result of DA is consistent.
(iv): A priority structure satisfies Ergin acyclicity.

Using these properties, we analyze the case where each school has a different preference order over students in the two-stage game.

## 3.3 Condition under which the two-stage game and DA have at least one equivalent outcome

Theorem 1 gives the condition sufficient to ensure that the two-stage game's SPE outcome is always the student-optimal stable matching. We now weaken Theorem 1's condition and show the condition sufficient for ensuring that at least one of the two-stage game's SPE outcomes is the student-optimal stable matching.

**Theorem 2.** *Suppose Ergin acyclicity holds, then there exists a matching $\mu$ which is the two-stage game's SPE outcome and satisfies*

$$\mu = \mu^{DA}.$$

Actually, the following strategy profile is an SPE of the two-stage game. In the first stage, each student $i$ applies to $\mu^{DA}(i)$. If $\mu^{DA}(i) = \emptyset$, $i$ does not apply to any school in the first stage. Under this strategy profile, the result of matching each student is the same as for DA matching, and thus the above theorem holds.

Note that under the Ergin acyclicity condition there can be SPE whose outcome is different from the matching generated by DA. Consider the following matching problem.

**Example 1.**

Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2\}, Q_{s_1} = Q_{s_2} = 2$, and $q_{s_1} = q_{s_2} = 1$. Consider the following preference structure.

$$\succ_{i_1}: s_1, s_2$$
$$\succ_{i_2}: s_2, s_1$$
$$\succ_{i_3}: s_1$$
$$\succ_{i_4}: s_2$$
$$\succ_{s_1}: i_3, i_4, i_2, i_1$$
$$\succ_{s_2}: i_4, i_3, i_1, i_2$$

It can be shown that there are no students and schools that satisfy Ergin's condition, and therefore Ergin acyclicity holds. It follows that

$$\mu^{DA}(i_1) = \mu^{DA}(i_3) = s_1$$
$$\mu^{DA}(i_2) = \mu^{DA}(i_4) = s_2.$$

In the two-stage game, the strategy profile under which $i_1$ applies to $s_2$, $i_2$ applies to $s_1$, and $i_3, i_4$ does not apply in the first stage is one of the SPEs. Under this strategy profile, $i_1$ and $i_4$ are matched to $s_2$, while $i_2$ and $i_3$ are matched to $s_1$. We now show that no students can gain by changing their strategy. Even if $i_1$ applies to $s_1$ or does not apply in the first stage, $s_1$ still accepts $i_2$ and $i_3$, and finally $i_1$ is matched to $s_2$ in the second stage. Similarly, $i_2$ cannot gain by changing his application in the first stage. $i_3$,$i_4$ cannot gain because they are matched to their best school. Therefore, this strategy profile is an SPE, and the matching is

$$\mu(i_2) = \mu(i_3) = s_1$$
$$\mu(i_1) = \mu(i_4) = s_2.$$

This is different from $\mu^{DA}$.

When Ergin acyclicity does not hold, in some cases, the SPE outcome in the two-stage game is always different from the matching generated by DA. Consider the following example.

**Example 2.**

Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3\}, Q_{s_1} = Q_{s_2} = Q_{s_3} = 1$, $q_{s_1} = q_{s_2} = 0$, and $q_{s_3} = 1$. Consider the following preference structure.

$$\begin{aligned}
&\succ_{i_1}: s_2, s_1, s_3 \\
&\succ_{i_2}: s_1, s_3 \\
&\succ_{i_3}: s_1, s_2, s_3 \\
&\succ_{i_4}: s_3 \\
&\succ_{s_1}: i_1, i_2, i_3 \\
&\succ_{s_2}: i_3, i_1 \\
&\succ_{s_3}: i_2, i_4
\end{aligned}$$

$(i_1, i_2, i_3)$ and $(s_1, s_2)$ satisfies the Cycle condition and the Scarcity condition since

$$i_1 \succ_{s_2} i_2 \succ_{s_1} i_3 \succ_{s_2} i_1$$

and there exists $I_{s_1}$ and $I_{s_2}$ such that

$$I_{s_1} = I_{s_2} = \emptyset.$$

Hence, Ergin acyclicity does not hold in this example. In this example, the matching generated by DA is

$$\begin{aligned}
\mu^{DA}(i_1) &= s_1 \\
\mu^{DA}(i_2) &= s_3 \\
\mu^{DA}(i_3) &= s_2 \\
\mu^{DA}(i_4) &= \emptyset.
\end{aligned}$$

We now consider SPEs of the two-stage game and show that their outcome is different from $\mu^{DA}$. First of all, note that students can apply to only $s_3$ in the first stage, and $i_1$ and $i_3$ are not acceptable to $s_3$. Hence, $i_1$ and $i_3$ are always matched to $s_1$ or $s_2$ in the second stage. Then, consider the strategies of $i_2$ and $i_4$. There are four cases.

*Case i :Both $i_2$ and $i_4$ apply to $s_3$ in the first stage.*

As a result, $i_2$ is matched to $s_3$ and $i_4$ is not matched to any school over the entire game. It is also the case that $i_4$ cannot be matched if $i_4$ does not apply in the first stage. If $i_2$ does not apply, $i_4$ is matched to $s_3$ and $i_2$ cannot be matched in the second stage.

*Case ii :Only $i_2$ applies to $s_3$ in the first stage.*

$i_2$ is matched to $s_3$ and $i_4$ is not matched. If $i_4$ applies to $s_3$, $i_4$ cannot be matched. If $i_2$ doesn't apply, $i_2$ is matched to $s_3$ in the second stage.

*Case iii :Only $i_4$ applies to $s_3$ in the first stage.*

$i_4$ is matched to $s_3$ and $i_2$ is not matched. If $i_2$ applies to $s_3$, $i_2$ will be matched

to $s_3$ in the first stage and gain benefit.
*Case iv :Neither $i_2$ nor $i_4$ applies to $s_3$ in the first stage.*
$i_2$ is matched to $s_3$ in the second stage and $i_4$ is not matched. If $i_4$ applies, $i_4$ can be matched to $s_3$ and gain benefit. In each of the SPEs in Cases i to iv, both $i_2$ and $i_4$ apply to $s_3$ or only $i_2$ applies to $s_3$. Then, both SPE results are

$$\mu(i_1) = s_2$$
$$\mu(i_2) = s_3$$
$$\mu(i_3) = s_1$$
$$\mu(i_4) = \emptyset.$$

This matching $\mu$ is different from $\mu^{DA}$, and $\mu$ Pareto dominates $\mu^{DA}$.

## 3.4 Condition under which the outcome of DA is weakly preferable to the two-stage game for every student

In Theorem 1 and Theorem 2, we consider what conditions in the two-stage game result in the student-optimal stable matching. Next, we analyze the relationship between the outcomes from a two-stage game and DA. We show the condition under which a change from a two-stage game to DA would be welcomed by students.

**Theorem 3.** *Let $\mu$, an arbitrary matching, be the SPE outcome of the two-stage game. Suppose the Cycle condition is not satisfied. Then, for all $i \in I$*

$$\mu^{DA}(i) \succsim_i \mu(i).$$

The proof is outlined as follows. Suppose $\mu(i_1) \succ_{i_1} \mu^{DA}(i_1)$ for some $i_1 \in I$. Then, there exists $i_2 \in I \backslash \{i_1\}$ such that

$$\mu^{DA}(i_2) = \mu(i_1) \text{ and } \mu(i_2) \neq \mu^{DA}(i_2).$$

We take such $i_2$ and prove that the Cycle condition holds in both of the following cases.

$$(Case\ i) : \mu^{DA}(i_2) \succ_{i_2} \mu(i_2)$$
$$(Case\ ii) : \mu(i_2) \succ_{i_2} \mu^{DA}(i_2)$$

In Case i, we show that $i_2$ is matched to $\mu(i_2)$ in the first stage of the two-stage game as the outcome of SPE, and consider the situation where only $i_2$ deviates from this and skips the first stage. Then, it can be shown that in the second stage of her deviating game, the Cycle condition holds true. In Case ii, if there exists a pair of students $(i, j)$ who satisfy

$$\begin{cases} \mu^{DA}(i) = \mu(j) \\ \mu^{DA}(i) \succ_i \mu(i) \\ \mu(j) \succ_j \mu^{DA}(j), \end{cases}$$

the Cycle condition is satisfied for the same reasons as in Case i. Otherwise, it can be shown that the outcome of DA is Pareto inefficient. Then, the priority structure is cyclical. (Ergin,2002). Therefore, the Cycle condition is self-evident.

If the Cycle condition is not satisfied, there exists no student who strictly prefers the two-stage game's SPE outcomes to the matching of DA. On the other hand, if the Cycle condition is satisfied and the Scarcity condition is not satisfied, there exist students who prefer the SPE outcomes to the matching of DA in some cases. Consider the following example.

**Example 3.**

Let $I = \{i_1, i_2, i_3, i_4, i_5\}$, $S = \{s_1, s_2, s_3, s_4\}, Q_{s_1} = Q_{s_2} = Q_{s_3} = 1$, $Q_{s_4} = 2$, $q_{s_2} = q_{s_3} = q_{s_4} = 1$, and $q_{s_1} = 0$. Consider the following preference structure.

$$\begin{aligned}
&\succ_{i_1}: s_1, s_2\\
&\succ_{i_2}: s_1, s_3, s_4\\
&\succ_{i_3}: s_4, s_1\\
&\succ_{i_4}: s_3\\
&\succ_{i_5}: s_4\\
&\succ_{s_1}: i_3, i_1, i_2, i_4, i_5\\
&\succ_{s_2}: i_3, i_1, i_2, i_4, i_5\\
&\succ_{s_3}: i_3, i_1, i_2, i_4, i_5\\
&\succ_{s_4}: i_2, i_3, i_1, i_4, i_5
\end{aligned}$$

$(i_1, i_2, i_3)$ and $(s_4, s_n)$ satisfy the Cycle condition for $n = 1, 2, 3$, but do not satisfy the Scarcity condition, since $i_3 \succ_{s_n} i_1 \succ_{s_n} i_2 \succ_{s_4} i_3$ and $i_2 \succ_{s_4} i_3 \succ_{s_4} i_1 \succ_{s_n} i_2$ but there does not exist $I_{s_4} \subset I \backslash \{i_1, i_2, i_3\}$ such that $I_{s_4} \subset U_{s_4}(i_3), |I_{s_4}| = Q_{s_4} - 1$. In this example, the matching generated by DA is

$$\begin{aligned}
\mu^{DA}(i_1) &= s_1\\
\mu^{DA}(i_2) &= s_3\\
\mu^{DA}(i_3) &= s_4\\
\mu^{DA}(i_4) &= \emptyset\\
\mu^{DA}(i_5) &= s_4.
\end{aligned}$$

In the two-stage game, when $i_1$ applies to $s_2$, $i_4$ applies to $s_3$, $i_5$ applies to $s_4$, and $i_2$ and $i_3$ do not apply to any schools in the first stage, the strategy profile is SPE. Then, the matching generated by the two-stage game is

$$\mu(i_1) = s_2$$
$$\mu(i_2) = s_1$$
$$\mu(i_3) = s_4$$
$$\mu(i_4) = s_3$$
$$\mu(i_5) = s_4,$$

and $i_2$ and $i_4$ prefer the outcome of this two-stage game to DA.

# 4 Discussion

So far, we have analyzed a two-stage game combining BM and DA in which students can choose whether to apply in the first stage. In this section, we consider how the results change when the game settings are slightly modified. If students must apply to one of the schools in the first stage, there is a counterexample to Theorem 1, Theorem 2, and Theorem 3. Consider the following example.

**Example 4.**

Let $I = \{i_1, i_2\}$, $S = \{s_1, s_2\}, Q_{s_1} = Q_{s_2} = 1$, $q_{s_1} = 0$, and $q_{s_2} = 1$. Consider the following preference structure.

$$\succ_{i_1}: s_1, s_2$$
$$\succ_{i_2}: s_1, s_2$$
$$\succ_{s_1}: i_1, i_2$$
$$\succ_{s_2}: i_1, i_2$$

In this example, the priority structure of each school over all students is the same as the other school's priority structure, the Cycle condition is not met, and Ergin acyclicity holds. The matching generated by DA is

$$\mu^{DA}(i_1) = s_1$$
$$\mu^{DA}(i_2) = s_2.$$

On the other hand, under the constraint that students have to apply to one of the schools in the first stage, the only outcome of the SPE of the two-stage game is

$$\mu(i_1) = s_2$$
$$\mu(i_2) = s_1.$$

For the above example, if students have to apply to one of the schools in the first stage, Theorem1, Theorem 2 and Theorem3 can be shown to be unsatisfied. In our analysis, therefore, the assumption that students can skip the first stage is important.

# 5 Concluding Remarks

We have discussed the characteristics of the subgame perfect equilibria outcomes of a two-stage game by comparing them to the agents-optimal stable matching. Our discussion has focused in particular on situations in which students apply for admission to schools. We first showed that if all schools have the same preference over the students, there is only one subgame perfect equilibrium outcome of the two-stage game, and it is equal to DA matching. To prove this, we considered an algorithm that selects which schools students go to in order of their grades. Second, we showed that when Ergin acyclicity is satisfied, at least one of the outcomes of the subgame perfect equilibria of the two-stage game accords with DA matching. On the other hand, if Ergin acyclicity is not satisfied, all subgame perfect equilibria outcomes of the two-stage game can be different from DA matching. Moreover, just because Ergin acyclicity is satisfied does not mean that all subgame perfect equilibria outcomes of the two-stage game coincide with DA matching. Finally, when the cycle condition, one of the conditions of Ergin cyclicity, does not hold, the outcome of DA is weakly preferable to the two-stage game's subgame perfect equilibria outcomes for every student. In addition, we showed that if students are forced to apply in the first stage, these theorems are not true. These results are valuable in analyzing how and the extent to which the two-stage game, when used in the real world, produces favorable outcomes for agents.

# Appendix: Proofs

## A: Proof of Theorem 1

Consider the matching generated by the following algorithm. Let $i_k \in I$ be the $k$-th best student for each school. Let $S^1 = S$ and $Q_s^1 = Q_s$ for each $s \in S$.
[***Algorithm 1***]
*Step 1:*
*Case i:* $i_1 \succ_s \emptyset$ *for each* $s \in S$.
Take $s_1 \in S^1$ which satisfies $s_1 \succsim_{i_1} s$ for all $s \in S^1$. Then, match $i_1$ with $s_1$, and let $Q_{s_1}^2 = Q_{s_1}^1 - 1$ and $Q_s^2 = Q_s^1$ for all $s \in S \backslash \{s_1\}$. If $Q_{s_1}^2 = 0$, let $S^2 = S^1 \backslash \{s_1\}$. Otherwise, let $S^2 = S^1$. Move to step 2.
*Case ii:* $\emptyset \succ_s i_1$ *for each* $s \in S$.
Let $s_1 = \emptyset$. $i_1$ cannot match with any school. Let $Q_s^2 = Q_s^1$ for all $s \in S$,and let $S^2 = S^1$. Move to step2.

*Step k* $(k = 2, 3, 4 \cdots)$*:*
*Case i:* $i_k \succ_s \emptyset$ *for each* $s \in S$.
Take $s_k \in S^k$ which satisfies $s_k \succsim_{i_k} s$ for all $s \in S^k$. Then, match $i_k$ with $s_k$, and let $Q_{s_k}^{k+1} = Q_{s_k}^k - 1$ and $Q_s^{k+1} = Q_s^k$ for all $s \in S \backslash \{s_k\}$. If $Q_{s_k}^{k+1} = 0$, let $S^{k+1} = S^k \backslash \{s_k\}$. Otherwise, let $S^{k+1} = S^k$. Move to step $k + 1$.
*Case ii:* $\emptyset \succ_s i_k$ *for each* $s \in S$.
Let $s_k = \emptyset$. $i_k$ cannot match with any school. Let $Q_s^{k+1} = Q_s^k$ for all $s \in S$, and let $S^{k+1} = S^k$. Move to step $k + 1$.

First, we prove $\mu(i_k) = s_k$ using mathematical induction.
$(i) : k = 1$
If $s_1 = \emptyset$, $\mu(i_1) = \emptyset$ because $s_1 = \emptyset$ is equivalent to $\emptyset \succ_s i_1$ for each $s \in S$. Now, suppose $s_1 \neq \emptyset$.
*Case i:* $q_{s_1} = 0$.
In this case, $i_1$ matches with $s_1$ by not applying to it in the first stage. Hence, $\mu(i_1) = s_1$
*Case ii:* $q_{s_1} > 0$.
In this case, an application by $i_1$ to $s_1$ in the first stage ensures that $i_1$ can match with $s_1$.
$(ii)$ : Suppose $\mu(i_1) = s_1, \mu(i_2) = s_2, \cdots, \mu(i_k) = s_k$. Then, prove that $\mu(i_{k+1}) = s_{k+1}$. If $s_{k+1} = \emptyset$, $\mu(i_{k+1}) = \emptyset$. Now, suppose $s_{k+1} \neq \emptyset$. Take any SPE strategy $a$ in the two-stage game. Then, we have

$$s_{k+1} \succsim_{i_{k+1}} \mu(i_{k+1}; a)$$

because $s_{k+1}$ is the best school for $i_{k+1}$ when conditioned on $\mu(i_1) = s_1, \mu(i_2) = s_2, \cdots, \mu(i_k) = s_k$.
*Case i:* $q_{s_{k+1}} > Q_{s_{k+1}} - Q_{s_{k+1}}^{k+1}$.
Suppose there exists an SPE strategy $a$ such that $\mu(i_{k+1}; a) \neq s_{k+1}$. Then, it

follows that

$$s_{k+1} \succ_{i_{k+1}} \mu(i_{k+1}; a).$$

However, since $\mu(i_1; a) = s_1, \mu(i_2; a) = s_2, \cdots, \mu(i_k; a) = s_k$, an application by $i_{k+1}$ to $s_{k+1}$ in the first stage ensures that $i_{k+1}$ can match with $s_{k+1}$, this is contradiction. Hence, we have

$$\mu(i_{k+1}) = s_{k+1}.$$

*Case ii:* $0 \leq q_{s_{k+1}} \leq Q_{s_{k+1}} - Q^{k+1}_{s_{k+1}}$.
Suppose there exists an SPE strategy $a$ such that $\mu(i_{k+1}; a) \neq s_{k+1}$. Then it follows that

$$s_{k+1} \succ_{i_{k+1}} \mu(i_{k+1}; a).$$

However, since $\mu(i_1; a) = s_1, \mu(i_2; a) = s_2, \cdots, \mu(i_k; a) = s_k$, $i_{k+1}$ can match with $s_{k+1}$ by not applying in the first stage, this is a contradiction. Therefore, it follows that

$$\mu(i_{k+1}) = s_{k+1}.$$

Second, we prove $\mu^{DA}(i_k) = s_k$. This can be proved by contradiction by supposing that $\mu^{DA}(i_k) \neq s_k$. Again, if $s_k = \emptyset$, we have $\mu^{DA}(i_k) = \emptyset$. Therefore, assume that $s_k \neq \emptyset$.
*Case i:*$\mu^{DA}(i_k) \succ_{i_k} s_k$.
In this case, there exists a student $i_A \in I$ who satisfies $i_A \succ_s i_k$ for each $s \in S$ who is matched with $\mu^{DA}(i_k)$ in algorithm 1 but who is not matched with $\mu^{DA}(i_k)$ in DA. If $\mu^{DA}(i_k) \succ_{i_A} \mu^{DA}(i_A)$, this contradicts the stability of DA because $(i_A, \mu^{DA}(i_k))$ can block $\mu^{DA}$. Hence, it follows that

$$\mu^{DA}(i_A) \succ_{i_A} \mu^{DA}(i_k).$$

Then, as noted above, there exists a student $i_B \in I$ who satisfies $i_B \succ_s i_A$ for each $s \in S$ who is matched with $\mu^{DA}(i_A)$ in algorithm 1 but who is matched with $\mu^{DA}(i_B)$ such that $\mu^{DA}(i_B) \succ_{i_B} \mu^{DA}(i_A)$ in DA. So there should exist an endless series of students $i_A, i_B, \cdots$ which satisfies

$$\cdots \succ_s i_B \succ_s i_A \succ_s i_k.$$

This contradicts the fact that the number of students is finite.
*Case ii:* $s_k \succ_{i_k} \mu^{DA}(i_k)$.
If $|\mu^{DA}(s_k)| < Q_{s_k}$,$(i_k, s_k)$ can block $\mu^{DA}$. Hence

$$|\mu^{DA}(s_k)| = Q_{s_k}.$$

We prove that there exists a student $i \in I \backslash \{i_k\}$ such that $\mu^{DA}(i) = s_k$ and $i_k \succ_s i$ for each $s \in S$. Since $s_k \in S^k$, $Q^k_{s_k} > 0$. Hence,

$$Q_{s_k} > Q_{s_k} - Q^k_{s_k}.$$

Therefore, the number of students who prefer $s_k$ and are preferred by $s_k$ over $i_k$ is less than $Q_{s_k}$. This is why there exists a student $i \in I\backslash\{i_k\}$ such that $\mu^{DA}(i) = s_k$ and $i_k \succ_s i$. Then, $(i_k, s_k)$ can block $\mu^{DA}$ because $i_k \succ_s i$ and $s_k \succ_{i_k} \mu^{DA}(i_k)$. This contradicts the stability of $\mu^{DA}$. This completes the proof of the statement that

$$\mu(i_k) = \mu^{DA}(i_k) = s_k$$

for all $k$ and all SPE outcomes $\mu$.

## B: Proof of Theorem 2

For each $i \in I$, let

$$a_i = \begin{cases} \mu^{DA}(i) & \text{If } \mu^{DA}(i) \neq \emptyset \text{ and } q_{\mu^{DA}(i)} > 0 \\ \emptyset & \text{Otherwise} \end{cases}$$

Let $a = (a_i)_{i \in I}$. Consider the strategy profile $a \in A$.

We first show that every student $i$ is matched to $\mu^{DA}(i)$ in this strategy profile when Ergin acyclicity holds. Note that each student $i$ applies to $\mu^{DA}(i)$ in the first stage unless they choose not to apply in the first stage. Hence, if $i$ is matched in the first stage, $i$ is matched to $\mu^{DA}(i)$. Since Ergin acyclicity implies consistency, other students, who are not matched to any school in the first stage, are also matched to the same school as in DA.

Next, we prove that this strategy profile $a \in A$ is an SPE of the two-stage game. If only student $i$ changes her application and is unmatched in the first stage, she is still matched to $\mu^{DA}(i)$ in the second stage because of consistency. Now, assume that student $i$ applies to school $s$ such that $s \succ_i \mu^{DA}(i)$ and $i$ is accepted in the first stage.

*Case i:* $|\mu^{DA}(s)| < q_s$.

In this case, $(i, s)$ can block $\mu^{DA}$ since $i$ is acceptable to $s$.

*Case ii:* $|\mu^{DA}(s)| \geq q_s$.

In this case, there exists $j \in \mu^{DA}(s)$ such that $i \succ_s j$. Hence, $(i, s)$ can block $\mu^{DA}$. Therefore, this contradicts the fact that $\mu^{DA}$ is a stable matching.

## C: Proof of Theorem 3

Take an SPE outcome matching $\mu$ of the two-stage game arbitrarily. Suppose that there exist students such that

$$\mu(i) \succ_i \mu^{DA}(i).$$

We can then show that the Cycle condition is satisfied. Let $i_1$ be such a student. Note that $i_1$ is rejected by $\mu(i_1)$ in DA and $i_1$ is acceptable to $\mu(i_1)$. Hence, the capacity of $\mu(i_1)$ is filled in DA. Thus, there exists a student $i_2 \in I\backslash\{i_1\}$ such that

$$\mu^{DA}(i_2) = \mu(i_1) \text{ and } \mu(i_2) \neq \mu^{DA}(i_2).$$

Take such $i_2$.
*Case i*:$\mu^{DA}(i_2) \succ_{i_2} \mu(i_2)$.
In this case, we have

$$i_2 \succ_{\mu^{DA}(i_2)} i_1$$

because if not, then $i_1 \succ_{\mu^{DA}(i_2)} i_2$ and $\mu^{DA}(i_2) = \mu(i_1) \succ_{i_1} \mu^{DA}(i_1)$ hold, which contradicts the stability of DA. If $i_1$ applies to $\mu(i_1)$ in the first stage and is accepted, $i_2$ can match with $\mu(i_1)$ by applying to it since $i_2 \succ_{\mu(i_1)} i_1$. However, this contradicts the fact that $i_2$ is matched with $\mu(i_2)$ as a result of the SPE, because $\mu(i_1) \succ_{i_2} \mu(i_2)$. Thus, $i_1$ is unmatched in the first stage and matched with $\mu(i_1)$ in the second stage. Also, the following argument shows that $i_2$ is matched in the first stage. If $i_2$ is matched with $\mu(i_2)$ in the second stage, it contradicts the stability of DA in the second stage because $i_2 \succ_{\mu(i_1)} i_1$ and $\mu(i_1) \succ_{i_2} \mu(i_2)$ hold. Therefore, $i_2$ matches with $\mu(i_2)$ in the first stage.
Consider a case in which only $i_2$ deviates from this SPE and chooses to skip the first stage. Let $\mu'$ be the matching generated in this case. Then,

$$\mu'(i_2) \neq \mu^{DA}(i_2) = \mu(i_1)$$

since $\mu^{DA}(i_2) \succ_{i_2} \mu(i_2)$ and $\mu$ is an SPE outcome. Also, it follows that

$$\mu'(i_1) \neq \mu(i_1)$$

because if $\mu'(i_1) = \mu(i_1)$, $i_2 \succ_{\mu^{DA}(i_2)} i_1$ and $\mu^{DA}(i_2) \succ_{i_2} \mu(i_2) \succsim_{i_2} \mu'(i_2)$ imply that $(i_2, \mu(i_1))$ can block $\mu'$. It is evident that there is at least one student $i_3$ who satisfies

$$i_3 \succ_{\mu(i_1)} i_2.$$

Next, we show there exist students such that

$$i \succ_{\mu(i_1)} i_2 \text{ and } \mu(i) \succ_i \mu'(i).$$

in the second stage in this SPE and the deviation case. Suppose, among students who join in that game, there is no such student. Then, for all $i$ such that $i \succ_{\mu(i_1)} i_2$,

$$\mu'(i) \succsim_i \mu(i)$$

is satisfied. $\mu(i_1) \succ_{i_2} \mu(i_2) \succsim_{i_2} \mu'(i_2)$ implies that for all $i^* \in \mu'(\mu(i_1))$,

$$i^* \succ_{\mu(i_1)} i_2.$$

If there exists $i^* \in \mu'(\mu(i_1))$ who satisfies $\mu(i_1) \succ_{i^*} \mu(i^*)$, $i^* \succ_{\mu(i_1)} i_2 \succ_{\mu(i_1)} i_1$ contradicts the the stability of DA in the second stage. Thus,

$$\mu(i^*) = \mu(i_1)$$

holds. This and $\mu'(i_1) \neq \mu(i_1)$ imply

$$|\mu'(\mu(i_1))| + 1 \leq |\mu(\mu(i_1))|.$$

$i_1$ is rejected by $\mu(i_1)$ when $i_2$ goes to the second stage, so there is no empty seat in $\mu(i_1)$ in the deviation case. This implies that the capacity of $\mu(i_1)$ in the second stage decreases when $i_2$ goes to the second stage. However, in this SPE, $i_2$ applies to $\mu(i_2)$ in the first stage, so the deviation increases the capacity of $\mu(i_2)$ in the second stage by one and does not change the capacity of $\mu(i_1)$ in the second stage. Thus, this is a contradiction. From this discussion, it is evident that among students who join in the second stage in this SPE and the deviation case, there is at least one student $i_3$ who satisfies

$$\begin{cases} i_3 \succ_{\mu(i_1)} i_2 \\ \mu(i_3) \succ_{i_3} \mu'(i_3) \\ \mu(i_3) \succsim_{i_3} \mu(i_1). \end{cases}$$

The last relationship holds because if $\mu(i_1) \succ_{i_3} \mu(i_3)$, it contradicts the stability of DA in the second stage. Next, we show that among students who join in the second stage in this SPE and the deviation case, there is at least one student $i_4$ who satisfies

$$\begin{cases} i_4 \succ_{\mu(i_3)} i_3 \\ \mu(i_4) \succ_{i_4} \mu'(i_4). \end{cases}$$

Suppose that among students who join in the second stage in this SPE and the deviation case, for all $i$ such that $i \succ_{\mu(i_3)} i_3$, $\mu'(i) \succsim_i \mu(i)$ is satisfied. Then, $\mu(i_3) \succ_{i_3} \mu'(i_3)$ implies that for all $i^* \in \mu'(\mu(i_3))$,

$$i^* \succ_{\mu(i_3)} i_3.$$

Therefore,

$$\mu(i_3) = \mu'(i^*) \succsim_{i^*} \mu(i^*)$$

holds for all $i^* \in \mu'(\mu(i_3))$. If there exists $i^*$ who satisfies $\mu(i_3) \succ_{i^*} \mu(i^*)$, $i^* \succ_{\mu(i_3)} i_3$ contradicts the stability of $\mu$. Thus, all $i^*$ satisfy

$$\mu(i^*) = \mu(i_3).$$

This and $\mu'(i_3) \neq \mu(i_3)$ imply

$$|\mu'(\mu(i_3))| + 1 \leq |\mu(\mu(i_3))|.$$

This is a contradiction for the same reasons as above. Moreover, as mentioned above, if $\mu(i_3) \succ_{i_4} \mu(i_4)$, $i_4 \succ_{\mu(i_3)} i_3$ contradicts the stability of DA in the second stage. Therefore, there is at least one student $i_4$ who satisfies

$$\begin{cases} i_4 \succ_{\mu(i_3)} i_3 \\ \mu(i_4) \succ_{i_4} \mu'(i_4) \\ \mu(i_4) \succsim_{i_4} \mu(i_3). \end{cases}$$

Continuing this discussion, we construct a sequence of students $\{i_1, i_3, i_4, \cdots\}$ which satisfies:

$$\begin{cases} i_3 \succ_{\mu(i_1)} i_1 \\ \mu(i_3) \succsim_{i_3} \mu(i_1) \\ i_1 \text{ joins in the second stage both in } \mu \text{ and } \mu' \\ i_{k+1} \succ_{\mu(i_k)} i_k \\ \mu(i_{k+1}) \succsim_{i_{k+1}} \mu(i_k) \\ i_k \text{ joins in the second stage both in } \mu \text{ and } \mu' \end{cases}$$

where $k = 3, 4, 5, \cdots$. This sequence continues endlessly. However, the number of students in the second stage is finite. Thus, there exists at least one student who satisfies $i_l = i_m$ $(l < m)$. Take such $l, m$. Note that $m \geq 4$ because $i_1 \neq i_3$. Then, it follows that

$$i_l \succ_{\mu(i_{m-1})} i_{m-1} \text{ and } \mu(i_l) \succsim_{i_l} \mu(i_{m-1}).$$

The former relationship holds because $i_l = i_m \succ_{\mu(i_{m-1})} i_{m-1}$. The later relationship holds because if not, $\mu(i_{m-1}) \succ_{i_l} \mu(i_l)$ is satisfied, which contradicts the stability of DA in the second stage.

Next, we show there exists $n \geq 4$ such that $i_1 = i_n$. Consider $i_{k+1}$ $(k = 3, 4, 5, \cdots)$ to be the student who is newly accepted or who has already been accepted to $\mu(i_k)$ in the step when $i_k$ is rejected by $\mu(i_k)$ in the deviation case. If there exists no $i_n \in I \backslash \{i_1\}$ such that $i_1 = i_n$, it follows that $l, m \in \{3, 4, 5, \cdots\}$ since it is assumed that $i_l = i_m$. In this case, the fact that $i_l$ is rejected by $\mu(i_l)$ means that $i_{l+1}$ is rejected by $\mu(i_{l+1})$ before $i_l$'s rejection from $\mu(i_l)$ in the deviation game. Then, the fact that $i_{l+1}$ is rejected by $\mu(i_{l+1})$ means that $i_{l+2}$ is rejected by $\mu(i_{l+2})$ before it in the deviation game. This chain finally implies the fact that $i_{m-1}$ is rejected by $\mu(i_{m-1})$ means that $i_m$ is rejected by $\mu(i_m)$ before it in the deviation game. This contradicts the statement that $i_l = i_m$. Therefore, there exists $n \geq 4$ such that $i_1 = i_n$.

We show, by mathematical induction, that if $n \geq 4$, there exists a set of three students and two schools that satisfies the Cycle condition.
(i):$n = 4$
In this case,

$$i_3 \succ_{\mu(i_1)} i_2 \succ_{\mu(i_1)} i_1 \succ_{\mu(i_3)} i_3$$

is satisfied. Hence, $\{i_1, i_2, i_3, \mu(i_1), \mu(i_3)\}$ satisfies the Cycle condition.
(ii):Assume that in the case of $n = k$, there exists a set of three students and two schools that satisfies the Cycle condition. Let us consider the case $n = k + 1$. Then it follows that

$$i_3 \succ_{\mu(i_1)} i_2 \succ_{\mu(i_1)} i_1 = i_{k+1} \succ_{\mu(i_k)} i_k \succ_{\mu(i_{k-1})} i_{k-1} \cdots i_5 \succ_{\mu(i_4)} i_4 \succ_{\mu(i_3)} i_3.$$

Note that $i_3 \neq i_5$. If $i_5 \succ_{\mu(i_3)} i_3$,

$$i_3 \succ_{\mu(i_1)} i_2 \succ_{\mu(i_1)} i_1 = i_{k+1} \succ_{\mu(i_k)} i_k \succ_{\mu(i_{k-1})} i_{k-1} \cdots i_5 \succ_{\mu(i_3)} i_3$$

is satisfied and this is the same case as $n = k$. If $i_3 \succ_{\mu(i_3)} i_5$, then

$$i_4 \succ_{\mu(i_3)} i_3 \succ_{\mu(i_3)} i_5 \succ_{\mu(i_4)} i_4$$

is satisfied, so $\{i_3, i_4, i_5, \mu(i_3), \mu(i_4)\}$ satisfies the Cycle condition. Therefore, the Cycle condition is satisfied if $\mu^{DA}(i_2) \succ_{i_2} \mu(i_2)$.

*Case ii:* $\mu(i_2) \succ_{i_2} \mu^{DA}(i_2)$.
In this case, $i_2$ is acceptable to $\mu(i_2)$, and its capacity is filled in DA because DA is individually rational. Thus, there exists at least one student who satisfies

$$i_3 \succ_{\mu(i_2)} i_2 \text{ and } \mu(i_2) = \mu^{DA}(i_3).$$

From the discussion above, if there exists a pair of students $(i, j)$ who meet

$$\begin{cases} \mu^{DA}(i) = \mu(j) \\ \mu^{DA}(i) \succ_i \mu(i) \\ \mu(j) \succ_j \mu^{DA}(j), \end{cases}$$

the Cycle condition is satisfied. Next, consider the situation that for all $i \in I$, there exists $j \in I$ who satisfies

$$\mu(j) \succ_j \mu^{DA}(j) = \mu(i) \succ_i \mu^{DA}(i).$$

Consider a chain of preference relations

$$\cdots = \mu(i_n) \succ_{i_n} \mu^{DA}(i_n) = \mu(i_{n-1}) \succ_{i_{n-1}} \mu^{DA}(i_{n-1}) \cdots \mu^{DA}(i_2) = \mu(i_1) \succ_{i_1} \mu^{DA}(i_1).$$

There exists a pair $(i_l, i_m)$ which satisfies $i_l = i_m$ and $l < m$ because there is a finite number of students. Then,

$$\begin{aligned} \mu(i_{l+1}) \succ_{i_{l+1}} \mu^{DA}(i_{l+1}) =& \mu(i_l) \succ_{i_l} \mu^{DA}(i_l) = \mu(i_{m-1}) \cdots \\ & \mu(i_{l+1}) \succ_{i_{l+1}} \mu^{DA}(i_{l+1}) = \mu(i_l) \succ_{i_l} \mu^{DA}(i_l) \end{aligned}$$

is satisfied. Now, we can assume that there exists no pair $(i_a, i_b)$ which satisfies $i_a = i_b$ and $l < a < b < m$. Let us define a new matching $\mu^{**}$ as below.

$$\mu^{**}(s) = \begin{cases} \mu(s) & (s \in \{s_l, s_{l+1}, \cdots, s_{m-1}\}) \\ \mu^{DA}(s) & (s \in S \backslash \{s_l, s_{l+1}, \cdots, s_{m-1}\}) \end{cases}$$

$$\mu^{**}(i) = \begin{cases} \mu(i) & (i \in \{i_l, i_{l+1}, \cdots, i_{m-1}\}) \\ \mu^{DA}(i) & (i \in I \backslash \{i_l, i_{l+1}, \cdots, i_{m-1}\}) \end{cases}$$

where $s_k = \mu(i_k)$. Since $\mu(i_l) = \mu^{DA}(i_{l+1}), \cdots, \mu(i_{m-1}) = \mu^{DA}(i_l)$, it follows that

$$|\mu^{**}(s)| = |\mu^{DA}(s)|$$

for all $s \in S$. Therefore, $\mu^{**}$ is feasible. It is evident that for all $i \in I$, $\mu^{**}(i) \succsim_i \mu^{DA}(i)$ and for $i \in \{i_l, i_{l+1}, \cdots, i_{m-1}\}, \mu^{**}(i) \succ_i \mu^{DA}(i)$. Therefore, DA is not Pareto efficient for students here, which means Ergin acyclicity is not satisfied per Ergin (2002), so the Cycle condition is satisfied.

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