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Abstract

This study compares standard auctions when bidders are financially constrained and valuation is endogenously determined by ex post investment. Bidders have a convex cost function because of financial constraints or borrowing costs. When the valuation is linear in investment, the revenue and investment equivalences hold. When the valuation is concave in investment, the first-price auction yields a higher expected revenue than the second-price auction, which is similar with the preceding models of financially constrained bidders. Although the ranking of investment and bidders' payoff is ambiguous, the second-price auction leads to larger expected investment and bidders' payoff under certain specifications. Scoring auctions can lead to larger investment while decreasing auction revenue.

Keywords: auction, financial constraint, investment, scoring auction
JEL Classification Codes: D44

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1 Introduction

The value created by a license, company, or business is highly dependent on its ex post investment. Although licenses and businesses are often sold or allocated through auctions, standard auction theory rarely deals explicitly with ex post investment.[1] This is because in a standard quasilinear model, the auction payment is sunk and does not affect ex post activities.[2] However, some empirical studies have shown that the introduction of spectrum auctions or higher spectrum prices had a negative impact on subsequent service diffusion or network coverage (Kuroda and Baquero, 2017; Bahia and Castells, 2022).

This phenomenon is consistent if bidders are financially constrained. If bidders face budget constraints, they may be forced to reduce their ex post investment when the amount paid in the auction soars. For high-stake auctions such as spectrum licenses and corporate takeovers, it is reasonable to suppose that bidders incur additional costs to finance a large auction payment, despite the absence of hard budget constraints.

The revenue equivalence theorem does not hold under financial or budget constraints. Che and Gale (1998) show that when bidders are financially constrained, a first-price auction (FPA) yields a higher expected revenue than a second-price auction (SPA). The intuition underlying the revenue ranking is the difference in bidders' incentives between auction rules. In the SPA, truthful bidding is a weakly dominant strategy, whereas bidders shade bids in the FPA in equilibrium. Hence, the budget constraint is weaker in the FPA than in the SPA, which leads to higher expected revenue.

However, the comparison of the seller's expected revenue is nontrivial in the presence of both financial constraints and ex post investment. In a situation when the value of the good is endogenously determined by ex post investment, the valuation is expected to decrease when the investment is suppressed under the high auction payment. Conversely, if an auction leads to a lower price, then ex post investment

[1] As exceptions, McAfee and McMillan (1986) and Laffont and Tirole (1987) study auctions of incentive contracts, in which winners take post-auction activities and moral hazard exists. Tomoeda (2019) shows that ex post investment is necessary to (fully) implement efficient outcomes in social choice problems including auctions.

[2] See Binmore and Klemperer (2002) for early debates on auctions and ex post investment in spectrum allocations.

will be stimulated and the valuation will increase. This would eventually increase auction revenue.

This study characterizes the revenue ranking between standard auctions when bidders are financially constrained and the value of the good is endogenously determined by ex post investment. We suppose that bidders face a soft financial constraint in which they incur a convex cost to finance auction payments and investment costs.[3] We show that if the valuation is linear in investment, revenue equivalence holds. Moreover, the expected investment is equivalent between the standard auction rules. Given the linear valuation and convex cost, the increase in the auction payment is completely offset by the decrease in investment. That is, the winner's total expenditure depends only on their type and not on the auction payment. The optimal investment and the winner's indirect utility are eventually linear in the auction payment. Thus, the winner's ex post payoff is reduced to quasilinear, and the revenue equivalence theorem holds. In addition, because the optimal investment is linear in auction payments, the expected investment is also equivalent.

We also show that if the valuation is concave in investment, the FPA provides a higher expected revenue than the SPA. When the valuation is concave in investment, the increase in the auction payment is not completely offset by the decrease in the optimal investment. Then, the winner's indirect payoff is concave in the auction payment, which is similar to risk-averse bidders. Therefore, well-known insights from auctions with risk-averse bidders, such as Maskin and Riley (1984) and Matthews (1987), are applied. The revenue ranking for the concave valuation is the same as in Che and Gale's (1998) financial constraint model and Baisa and Rabinovich's (2016) endogenous budget model. Our revenue ranking holds even if bidders face heterogeneous financial constraints which are their private information.

In contrast to the clear revenue ranking, comparisons regarding the investment and bidders' payoffs are ambiguous. When the valuation is concave in investment, we provide specifications under which the SPA enhances ex post investment. In addition, we give specifications under which bidders' expected payoff is greater in the SPA than in the FPA.

Due to the financial cost, standard auctions lead to under-investment in the sense that the winner's ex post investment is lower than when bidders are not financially

[3] Soft financial constraints or borrowing costs are studied in Che and Gale (1998), Rhodes-Kropf and Viswanathan (2005), and Baisa and Rabinovich (2016) among others.

constrained. Also, in the case of public license auctions, the seller (government) may want to promote investment from the viewpoint of consumer surplus. We examine beauty contests and scoring auctions for the promotion of investment. We argue that beauty contests may suffer from an over-investment. Scoring auctions, which lie between standard (price-only) auctions and beauty contests, may promote investment to the extent that it does not lead to over-investment.

2 Model

Suppose that a single good, such as a license or firm, is auctioned among $n \geq 2$ potential buyers. We consider standard auctions, mainly, first- and second-price sealed-bid auctions. After auction, the winner makes an investment that determines the valuation of the good. Bidder i's valuation is denoted by $V(\theta_i, x)$, where $x \in \mathbb{R}_+$ is the post-auction investment and $\theta_i \in \Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$ is i's private type. Each bidder's type θ_i is independently and identically distributed with a cumulative distribution F. We assume that F has a density function $f = F' > 0$ on Θ. We suppose that the valuation function $V : [\underline{\theta}, \bar{\theta}] \times \mathbb{R}_+ \to \mathbb{R}_+$ is identical for all bidders. We also assume that V is thrice differentiable and strictly increasing in both θ_i and x, and that the single crossing condition $V_{x\theta} \equiv \partial^2 V / \partial x \partial \theta \geq 0$ holds.

Bidders are financially constrained and incur a common convex cost to finance auction fees and investments.[4] The cost function is denoted by $C : \mathbb{R} \to \mathbb{R}$. It is thrice differentiable, strictly convex, and satisfies $C(0) = 0$, $c \equiv C' \geq 1$, and $c' \equiv C'' > 0$. When bidder i wins the auction with a payment p and makes an ex post investment x, their payoff is given by[5]

$$u(\theta_i, p, x) \equiv V(\theta_i, x) - C(p + x).$$

We suppose that when a bidder loses the auction, they do not invest and their payoff is zero.

We assume that the winner's payoff u is concave in x and that there is a unique optimal investment.

Assumption 1 Suppose that $u(\theta_i, p, x)$ is concave in x for all $\theta_i \in \Theta$ and all p (in equilibrium) and that $\max_x u(\theta_i, p, x)$ has a unique interior solution $x^* \in (0, \infty)$.

[4] We introduce bidders' private information on their cost functions in section 4.

[5] The singular pronoun "they" is used for gender-neutral language.

Given that bidder i wins the auction by paying p, the optimal investment is denoted by

$$x^*(\theta_i, p) \in \arg\max_x u(\theta_i, p, x). \tag{1}$$

Given the optimal investment x^*, the winner's ex post payoff is reduced to

$$U(\theta_i, p) \equiv u\left(\theta_i, p, x^*(\theta_i, p)\right). \tag{2}$$

Note that U satisfies the single crossing condition. Hence, there exists a pure strategy Bayesian Nash equilibrium in standard auctions (Athey, 2001).

Note that if bidders are not financially constrained and $C(y) = y$, the winner's optimal investment is independent of their auction payment. The model is reduced to a standard quasilinear model with the valuation function $V^*(\theta_i)$, where

$$V^*(\theta_i) \equiv \max_x V(\theta_i, x) - x. \tag{3}$$

Hence, the revenue equivalence theorem holds: the seller's expected revenue is equivalent between the FPA, SPA, and any other standard auctions in which the bidder of the highest type always wins and the lowest-type's payoff is zero. In addition, post-auction investment is ex post equivalent between such auction rules.

3 Main Result

We first characterize the equilibrium of SPA and FPA. Let $w(\theta_i)$ be the unique solution of

$$U(\theta_i, w(\theta_i)) = 0.$$

That is, w is the auction payment such that the winner's ex post payoff is zero. It is straightforward to confirm that w is increasing in θ_i.

Note that $w(\theta_i)$ is the maximum willingness to pay of bidder i of type θ_i in the auction. In the SPA, it is a weakly dominant strategy to bid $w(\theta_i)$ as in the standard quasilinear model. The following lemma is shown in a similar manner to the quasilinear case. Similar results are provided by Maskin and Riley (1984), Sakai (2008), and Saitoh and Serizawa (2008); thus, the proof is omitted.

Lemma 1 *In the SPA, it is a weakly dominant strategy for each bidder to submit* $w(\theta_i)$.

It is difficult to obtain an explicit solution for the FPA. The equilibrium bidding function in the FPA is characterized by the first-order condition as follows. Let $G = F^{n-1}$ be the distribution of the largest order statistic of $n-1$ independent draws from F.

Lemma 2 *The symmetric Bayesian Nash equilibrium bidding function β in the FPA is given by*

$$\beta'(\theta_i) = -\frac{U(\theta_i, \beta(\theta_i))}{U_p(\theta_i, \beta(\theta_i))} \cdot \frac{g(\theta_i)}{G(\theta_i)} \tag{4}$$

with $\beta(\underline{\theta}) = w(\underline{\theta})$.

Proof See Appendix.

3.1 Revenue Ranking

The revenue ranking between the FPA and SPA depends on the shape of the winner's indirect utility U. The shape of U is critically determined by the concavity or convexity of the valuation function V. Our main result is stated as follows.

Theorem 1 *The following statements hold.*

1. *If V is concave in x, the expected revenue of the FPA is greater than that of the SPA.*

2. *If V is convex in x, the expected revenue of the SPA is greater than that of the FPA.*

3. *If V is affine in x, the expected revenue and expected investment are equivalent between the FPA and SPA.*

Proof Suppose that V is concave in x. The first-order condition for the optimal investment is

$$V_x(\theta_i, x^*) = c(p + x^*).$$

By differentiation of the implicit function, we have

$$-x_p^*(\theta_i, p) = \frac{c'(p + x^*)}{c'(p + x^*) - V_{xx}(\theta_i, x^*)} \leq 1.$$

By the envelope theorem, we have

$$U_p(\theta_i, p) = -c\left(p + x^*(\theta_i, p)\right), \tag{5}$$

and

$$U_{pp}(\theta_i, p) = -\left(1 + x_p^*(\theta_i, p)\right) c'\left(p + x^*(\theta_i, p)\right) \leq 0. \tag{6}$$

Hence, $U(\theta_i, p)$ is concave in p. Then, we have the first statement by applying Theorem 4 of Maskin and Riley (1984).

The other statements are analogous to the first statement. When V is affine in x, the equivalence in the expected investment holds because the revenue equivalence holds and the optimal investment is linear in the auction payment, $x_p^*(\theta_i, p) = -1$ for all p. ■

The marginal increase in auction payments decreases the ex post investment because of the convex cost. When the valuation function V is concave, the marginal decrease in investment does not completely offset the increase in auction payment. Hence, the winner's total expenditure is increasing in auction payment. Due to the convex cost function, this means that the marginal cost of the winner's payoff is increasing in auction payment. Therefore, the same revenue ranking with the models of financially constrained bidders, such as Che and Gale (1998) and Baisa and Rabinovich (2016), holds.

When valuation function is affine in x, the marginal increase in the auction payment is completely offset by the decrease in investment. The winner's total expenditure is constant and depends only on their own type. With linear valuation, the winner's indirect payoff is eventually linear in the auction payment, and the model is reduced to quasilinear.

Example 1 Suppose $V(\theta_i, x) = \theta_i x$. Under this specification, the first-order condition for the optimal investment yields

$$\theta_i - c(p + x) = 0.$$

Hence,

$$x^*(\theta_i, p) = c^{-1}(\theta_i) - p.$$

The winner's indirect utility U is given by

$$\begin{aligned} U(\theta_i, p) &= \theta_i\left(c^{-1}(\theta_i) - p\right) - C\left(c^{-1}(\theta_i)\right) \\ &= \theta_i\left(w(\theta_i) - p\right), \end{aligned}$$

where

$$w(\theta_i) = c^{-1}(\theta_i) - \frac{C\left(c^{-1}(\theta_i)\right)}{\theta_i}.$$

Note that each bidder's type θ_i is a given parameter in the interim stage of the auction. Hence, the model is indeed equivalent to a standard quasilinear model with the valuation function w. Therefore, the revenue equivalence theorem holds. Moreover, because the optimal investment is linear in the auction payment, the expected post-auction investment is equivalent between standard auction rules.

3.2 Investment and Winner's Payoff

When valuation function is affine, our model is reduced to a quasilinear model. Hence, in addition to the expected revenue for the seller, equivalences in terms of expected investment and bidders' payoffs holds between standard auctions.

However, in cases of concave or convex valuations, the rankings of the winner's investment and payoff levels are ambiguous. In the FPA, the winner's ex post investment is deterministic in the bidding stage. However, in the SPA, the ex post investment is probabilistic because the auction payment is uncertain in the interim. When the winner's payoff is concave in the auction payment, the increase in the expected auction payment may not imply that the FPA is worse for bidders than the "risky" SPA.

In the rest of this section, we suppose that V is (strictly) concave, which would be the most realistic. Thus, the expected revenue of the FPA is greater than that of the SPA. The ranking of the winner's investment depends on the shape of the optimal investment x^*. The following result provides a condition under which the SPA provides a greater expected investment.

Proposition 1 *Suppose $V_{xx} < 0$, $V_{xxx} \geq 0$ and $c'' \leq 0$. Then, the optimal investment x^* is convex in p, and the expected investment after the SPA is greater than that after the FPA.*

Proof See Appendix.

For example, the assumptions in Proposition 1 hold when

$$V(\theta_i, x) = \theta_i x^\alpha$$

with $\alpha \in (0, 1)$ and

$$C(y) = y + \delta y^{\gamma}$$

with $\gamma \in (1, 2]$ and $\delta > 0$. The logarithmic valuation $V(\theta_i, x) = \theta_i \log x$ also induces $V_{xxx} \geq 0$.

Matthews (1987) compares the risk-averse bidders' payoffs in standard auctions. He shows that bidders' expected payoff is greater in the SPA than in the FPA if winner's indirect utility U satisfies "decreasing absolute risk aversion (DARA)": $\partial(U_{pp}/U_p)/\partial\theta < 0$. Although we do not have a concise condition under which U possesses DARA property, the following specification induces DARA.

Proposition 2 *If $c'' \leq 0$ and $x^*_{p\theta} \leq 0$, then the winner's indirect utility U possesses DARA property, and each bidder's expected payoff is greater in the SPA than in the FPA. The winner's indirect utility U possesses DARA property when V is strictly concave, $V_{xxx} \geq 0$, $V_{xx\theta} \geq 0$, and the cost function $C(y) = y + \delta y^2$ with $\delta > 0$.*

Proof See Appendix.

Valuation functions that induce DARA include $V(\theta_i, x) = x^{\alpha} + \theta_i x$ with $\alpha < 1$ and $V(\theta_i, x) = (A + \theta_i)x - x^2$ with a sufficiently large constant $A > 0$.

4 Heterogeneity in Financial Costs

Thus far, we have assumed that bidders face the same financial constraint C. In reality, bidders usually face different financial constraints, which are their private information. Our main result still holds even if cost functions are different between bidders and private information. This is confirmed in a similar manner with Che and Gale (1998).

In this section, we incorporate bidders' additional private information into financial constraints. Bidder i faces a cost function $C(y \mid \omega_i)$, where $\omega_i \in \Omega \equiv [\underline{\omega}, \bar{\omega}] \subset \mathbb{R}_+$ is the bidder's cost type. Let ω^* represent an unconstrained bidder: $C(y \mid \omega^*) = y$ for all $y \geq 0$. Suppose that $C(\cdot \mid \omega_i)$ is strictly convex for all $\omega_i < \omega^*$. Suppose that the largest cost type $\bar{\omega} \leq \omega^*$ and that an unconstrained type may not exist in Ω. For simplicity, we assume that C is continuously differentiable in ω_i, $\partial C(y \mid \omega_i)/\partial \omega_i \leq 0$, and $\partial^2 C/(\partial y \partial \omega_i) \leq 0$. That is, a large cost type represents a less constrained bidder, and the marginal cost $c(\cdot \mid \omega_i)$ is non-increasing in ω_i.

Each bidder's type is a pair of their valuation type θ_i and cost type ω_i and denoted by $t_i = (\theta_i, \omega_i) \in T \equiv \Theta \times \Omega$. Their winning payoff is denoted by

$$u(t_i, p, x) = V(\theta_i, x) - C(p + x \mid \omega_i).$$

The winner's indirect utility is reduced to $U(t_i, p) \equiv \max_x u(t_i, p, x)$, and the optimal investment is denoted by $x^*(t_i, p) \in \arg\max u(t_i, p, x)$. Similar to the previous sections, we assume that the optimal investment problem has a unique interior solution for all t_i in equilibrium.

Assumption 2 Suppose that $u(t_i, p, x)$ is concave in x for all $t_i \in T$ and all p (in equilibrium) and that $\max_x u(t_i, p, x)$ has a unique interior solution $x^* \in (0, \infty)$.

The maximum willingness to pay $w(t_i)$ of bidder i of type t_i is implicitly defined by

$$U\left(t_i, w(t_i)\right) = 0.$$

Because the cost type ω^* is unconstrained, we have

$$w(\theta_i, \omega^*) = V^*(\theta_i) = \max_x V(\theta_i, x) - x$$

for all $\theta_i \in \Theta$.

4.1 When Valuation Is Affine in Investment

Consider the case where valuation is affine in investment. Let $\lambda(\theta_i) \equiv V_x(\theta_i, x) > 0$. Because the optimal investment x^* needs to exist in the interior, we assume that bidders are necessarily financially constrained and $\bar{\omega} < \omega^*$. Each bidder's type t_i is independently and identically distributed over T with a cumulative distribution F.[6]

By the analysis thus far, a winner's indirect payoff function is reduced to

$$U(t_i, p) = \lambda(\theta_i) \left(w(t_i) - p\right).$$

Let $T(v) \equiv \{t_i \in T \mid w(t_i) \leq v\}$, which is the set of types such that the associated willingness to pay is equal to or less than v. In addition, let $H(v) \equiv \Pr\{t_i \in T(v)\}$ be the cumulative distribution of a bidder's willingness to pay. Our model is equivalent to the quasilinear model with valuation v_i, which is drawn from the distribution

[6] We allow that θ_i and ω_i are correlated. However, types are independent across bidders.

H. Therefore, we have the revenue equivalence theorem for standard auctions. In equilibrium, the bidder with the largest willingness to pay $w(t_i)$ wins. In addition, the optimal investment x^* is linear in p, thus that the expected investment is also equivalent between standard auctions.

Proposition 3 *Suppose that bidders are necessarily financially constrained, $\bar{\omega} < \omega^*$, and that Assumption 2 holds. If V is affine in x, then the expected revenue and expected investment are equivalent between the FPA and SPA.*

4.2 When Valuation Is Concave in Investment

Consider the case where valuation is strictly concave in investment. In this case, we suppose that bidders can be unconstrained and $\bar{\omega} = \omega^*$.

Consider the SPA. Analogous to the previous section, each bidder has a weakly dominant strategy to submit $w(t_i)$. Let $T^{II}(v)$ be the set of all types who submit a bid equal or less than v. That is, $T^{II}(v) = \{t_i \in T \mid w(t_i) \leq v\}$. Let $H^{II}(v) = \Pr\{t_i \in T^{II}(v)\}$ be the cumulative distribution of a bidder's willingness to pay. It is clear that the seller's revenue is equivalent to the quasilinear model with valuation v_i which is independently drawn from H^{II}.

Consider the FPA. Suppose that there exists a symmetric Bayesian Nash equilibrium $\beta(t_i)$, which is increasing in both θ_i and ω_i. Suppose that all bidders except for bidder i take the equilibrium strategy β. Let Q be the cumulative distribution of the highest bid of all bidders except for i in equilibrium, and let $q = Q'$ be its density.

Consider bidder i of type $t_i = (\theta_i, \omega^*)$. Because bidder i is unconstrained, their expected payoff of bidding b is given by

$$Q(b)\left(V^*(\theta_i) - b\right).$$

The first-order condition for the equilibrium bid $b^* = \beta(\theta_i, \omega^*)$ is given by

$$\begin{aligned} q(b^*)(V^*(\theta_i) - b^*) - Q(b^*) &= 0 \\ \therefore \quad V^*(\theta_i) - b^* &= \frac{Q(b^*)}{q(b^*)}. \end{aligned} \tag{7}$$

Consider an arbitrary type $\tilde{t}_i = (\tilde{\theta}_i, \tilde{\omega}_i)$ such that the associated equilibrium bid is

$\beta(\tilde{t}_i) \le b^*$. Then, the marginal payoff of bidding b^* is nonpositive and

$$q(b^*)U(\tilde{t}_i, b^*) + Q(b^*)U_p(\tilde{t}_i, b^*) \le 0$$
$$\therefore \quad -\frac{U(\tilde{t}_i, b^*)}{U_p(\tilde{t}_i, b^*)} \le \frac{Q(b^*)}{q(b^*)} = V^*(\theta_i) - b^* \tag{8}$$

by (7). The inequality (8) specifies the set $T^I(v) \equiv \{\tilde{t}_i \in T \mid \beta(\tilde{t}_i) \le b^* = \beta(\theta_i, \omega^*)\}$ with $v = V^*(\theta_i)$. That is, $T^I(v)$ is the set of types who lose to an unconstrained bidder with valuation $v = V^*(\theta_i)$. Let $H^I(v) \equiv \Pr\{t_i \in T^I(v)\}$ be the cumulative distribution of the valuation v. Bidders behave in equilibrium as if they have a quasilinear payoff with some valuation v_i which is independently distributed by H^I. The seller's revenue is equivalent to the quasilinear model with valuation v_i which is independently drawn from H^I.

It is obvious that $w(\tilde{t}_i) \le b^*$ implies $w(\tilde{t}_i) \le V^*(\theta_i)$. Suppose $w(\tilde{t}_i) > b^*$. By strict concavity of U, we have

$$-U_p(\tilde{t}_i, b^*) < \frac{U(\tilde{t}_i, b^*)}{w(\tilde{t}_i) - b^*}. \tag{9}$$

(8) and (9) imply $w(\tilde{t}_i) < V^*(\theta_i)$. Therefore, we have for all $\theta_i \in \Theta$ and $v = V^*(\theta_i)$,

$$\tilde{t}_i \in T^I(v) \Rightarrow \tilde{t}_i \in T^{II}(v).$$

That is, $H^I(v) \le H^{II}(v)$ for all v. Because (9) is strict inequality, the distribution H^I stochastically dominates H^{II}. The seller's expected revenues in FPA and SPA are the expected second-highest order statistic from distributions H^I and H^{II}, respectively. Hence, the FPA provides a greater expected revenue for the seller than the SPA.

Proposition 4 *Suppose that bidders' cost type space satisfies $\bar{\omega} = \omega^*$, and that Assumption 2 holds. If V is strictly concave in x, the expected revenue of the FPA is greater than that of the SPA.*

5 Investment Promotion Policy

Due to financial costs, standard auctions lead to "under-investment" in the sense that

$$x^*(\theta_i, p) < \hat{x}(\theta_i),$$

where

$$\hat{x}(\theta_i) \in \arg\max_x V(\theta_i, x) - x$$

is the optimal investment for type θ_i when there is no financial constraint on the bidder. In addition, in license auctions conducted by governments, policymakers may want to implement a higher level of investment to increase consumers' surplus and social welfare. A simple solution to the underinvestment is to provide a sufficiently large subsidy to each bidder and eliminate financial constraints. However, such a policy would be highly costly and unrealistic.

Beauty contests and scoring auctions are alternatives without the subsidization of investment promotion. We argue that beauty contests may suffer from over-investment. Scoring auctions, which can be considered as an intermediate between standard auctions and beauty contests, can promote investment to the extent that it does not lead to over-investment.

5.1 Beauty Contests

Despite the global trend of using auctions to allocate spectrum licenses, Japan is unique among developed countries, as it allocates spectrum licenses through beauty contests or comparative hearing rather than auctions. A typical dimension of the contest for Japanese spectrum allocations is the commitments to future investments and service quality such as coverage. Such a beauty contest may be considered as a simple fixed-price best proposal auction (Thiel, 1988).

Assume that the licenser (government) collects only the minimum license fee $\underline{p}$. In a beauty contest, each bidder submits a proposal that commits an ex post investment x_i. The bidder who submits the highest proposal wins. Consider the "first-score beauty contest," in which the winner needs to invest at least as much as their own proposal. In the beauty contest, the winner's ex post payoff when submitting a proposal x_i is given by

$$U^{BC}(\theta_i, x_i) = \begin{cases} U(\theta_i, \underline{p}) = V(\theta_i, x^*(\theta_i, \underline{p})) - C(\underline{p} + x^*(\theta_i, \underline{p})) & \text{if } x_i < x^*(\theta_i, \underline{p}) \\ V(\theta_i, x_i) - C(\underline{p} + x_i) & \text{if } x_i \geq x^*(\theta_i, \underline{p}) \end{cases}.$$

Note that bidders have no incentive to submit x_i smaller than the optimal investment x^*. They voluntarily invest at least the optimal investment level $x^*(\theta_i, \underline{p})$. The winner's payoff U^{BC} is non-increasing and concave in x_i by assumption. Although the competition in terms of price turns to that in terms of investment, the equilibrium analysis is similar to the previous section and Maskin and Riley (1984).

In the beauty contest, the winner makes a large investment. Any proposal $x_i < x^*(\theta_i, \underline{p})$ is dominated by $x^*(\theta_i, \underline{p})$. It is clear that the equilibrium proposal $\beta^{BC}(\theta_i)$ in the first-score beauty contest satisfies

$$\beta^{BC}(\theta_i) \geq x^*(\theta_i, \underline{p}).$$

In contrast to standard auctions, the beauty contests may lead to "over-investment." To see this, it would be helpful to consider the "second-score beauty contest," in which the winner needs to invest at least as much as the second highest proposal.[7] Analogous to the SPA, there exists a weakly dominant strategy in the second-score beauty contest, $w^{BC}(\theta_i)$, which is implicitly determined by

$$U^{BC}(\theta_i, w^{BC}(\theta_i)) = 0.$$

Bidders propose such a large investment that they earn zero profits. Of course, the winner's actual investment is smaller than their proposal generically. However, if the contest is sufficiently competitive and there are many bidders, the winner's ex post investment would be close to the zero-profit level with a high probability. Moreover, because U^{BC} is concave in x_i, the expected investment in equilibrium is larger in the first-score beauty contest than in the second-score beauty contest. Although we do not define the socially optimal investment or what is the over-investment, the equilibrium property here would suggest that the equilibrium investment is often excessive for policymakers.

5.2 Scoring Auctions

Another alternative is scoring auctions (Che, 1993; Asker and Cantillon, 2008). Scoring auctions evaluate both quality (investment) and price, and is an intermediate mechanism between standard (price-only) auctions and beauty contests. Consider a scoring auction in which the scoring function is quasilinear and given by $S(x, p) = q(x) + p$, where q is strictly increasing and normalized by $q(0) = 0$.[8] Each

[7] In the second-score beauty contest, the winner's investment needs not to be equal to the second highest proposal, as the winner may have an incentive to invest more when the second highest proposal is too small.

[8] See Hanazono et al. (2020) for general scoring rules including non-quasilinear scoring functions.

bidder submits a proposal (x_i, p_i) with $x_i \geq 0$ and $p_i \geq 0$, and the bidder of the highest score $S(x_i, p_i)$ wins. We consider both first- and second-score auctions. In the first-score auction, the winner's own proposal is enforced. In the second-score auction, the winner can implement any investment-price pair (x_i, p_i) that fulfills the second-highest score.

The Bayesian Nash equilibrium of the scoring auctions is derived in a similar manner with Che (1993). When a contract with a score of $s = q(x) + p$ is enforced for the winner, their ex post payoff is denoted by

$$\hat{u}(\theta_i, s, x) \equiv V(\theta_i, x) - C\left(s - q(x) + x\right).$$

Let

$$\hat{U}(\theta_i, s) = \max_{x|0 \leq q(x) \leq s} \hat{u}(\theta_i, s, x)$$

be the maximum payoff when a contract with score s is enforced. In addition, let

$$\hat{x}^*(\theta_i, s) \in \arg \max_{x|0 \leq q(x) \leq s} \hat{u}(\theta_i, s, x)$$

be the optimal investment proposal subject to score s. The Bayesian Nash equilibrium is derived by solving the associated auctions in terms of the score bid s, given that bidders propose the investment bid $\hat{x}^*(\theta_i, s)$.

In the rest of this section, we suppose that V is concave. In addition, we focus on scoring function such that the virtual cost function $C(s - q(x) + x)$ is strictly convex.

Assumption 3 The function $\hat{C}(x, s) = C(s-q(x)+x)$ is strictly increasing and convex in x for all s. In addition, the optimal investment problem $\max_{x|0 \leq q(x) \leq s} \hat{u}(\theta_i, s, x)$ has a unique interior solution in equilibrium.

Assumption 3 implies $q'(x) < 1$ for all x.

For the winner, the optimal investment $\hat{x}^*$ after the scoring auction is given by the first-order condition

$$V_x(\theta_i, \hat{x}^*) = \left(1 - q'(\hat{x}^*)\right) c\left(s - q(\hat{x}^*) + \hat{x}^*\right). \tag{10}$$

Note that $s - q(\hat{x}^*) + \hat{x}^* = p + \hat{x}^*$. Hence, the first-order condition (10) is written as

$$p + \hat{x}^* = c^{-1}\left(\frac{V_x}{1 - q'(\hat{x}^*)}\right). \tag{11}$$

In contrast, the first-order condition for the optimal investment x^* after a standard (price-only) auction yields

$$p + x^* = c^{-1}\left(V_x\right).$$

Therefore, for any given auction payment p, the winner makes a larger investment in scoring auctions than in standard auctions. As bidders commit to larger investments in scoring auctions, their ex post payoffs are smaller than those in standard auctions. Therefore, the equilibrium price bid decreases in scoring auctions, further enhancing the investment incentive.

Proposition 5 *Suppose that V is concave and that Assumption 3 holds. If the scoring auctions have a symmetric increasing equilibrium, then the ex post investments in the first- and second-score auctions are greater than those in the first- and second-price standard auctions, respectively. In addition, the seller's revenues in the first- and second-score auctions are lower than those in the first- and second-price standard auctions, respectively.*

Proof See Appendix.

Notice that scoring auctions with $q'(x) \geq 1$ lead to beauty contests. Suppose that $q'(x) \geq 1$ for all x. The marginal payoff, given a winning score s, is

$$V_x(\theta_i, x) - (1 - q'(x))c(s - q(x) + x) > 0$$

for all x. Hence, given non-negative payment $p_i \geq 0$, the optimal price bid in the scoring auction is zero for all θ_i: the scoring auction is equivalent to a beauty contest. This is because if $q' \geq 1$, the marginal increase in investment always has a greater effect on the score than the price bid. Hence, bidders have no incentive to compete in terms of the price bid. As such, scoring auctions with high allocation of the investment component are equivalent to beauty contests.

6 Conclusion

We have compared standard auctions with financially constrained bidders when the valuation is endogenously determined by the winner's ex post investment. The revenue ranking between the FPA and SPA depends on the shape of the valuation function. If the valuation is concave in investment, the FPA provides a higher expected revenue for the seller than the SPA, which is similar to the standard model of financially constrained bidders. Interestingly, if the valuation is linear in investment, the equivalence theorem holds for both expected revenue and investment. This result holds even if bidders have different financial constraints which are private information of bidders.

We have also considered beauty contests and scoring auctions for promoting investment. Scoring auctions with quasilinear scoring rules induce higher ex post investments with decreasing auction revenue. When policymakers want to increase ex post investment rather than auction revenue in license auctions, scoring auctions can be an alternative.

A Proofs

A.1 Proof of Lemma 2

Suppose that there exists a symmetric Bayesian Nash equilibrium β, and that every bidder other than i follows the equilibrium strategy. The interim expected payoff when bidder i makes an equilibrium bid of type z is

$$\pi_i(z, \theta_i) = G(z) U\left(\theta_i, \beta(z)\right),$$

where $G = F^{n-1}$ is the distribution of the largest order statistic of $n-1$ independent draws from F. The first-order condition for the payoff maximization is

$$g(z) U\left(\theta_i, \beta(z)\right) + \beta'(z) G(z) U_p\left(\theta_i, \beta(z)\right) = 0,$$

where $g = G'$. Because the first-order condition should hold with $z = \theta_i$, we have

$$g(\theta_i) U\left(\theta_i, \beta(\theta_i)\right) + \beta'(\theta_i) G(\theta) U_p\left(\theta_i, \beta(\theta_i)\right) = 0.$$

The terminal condition for the differential equation is $U(\underline{\theta}, \beta(\underline{\theta})) = 0$. Thus, $\beta(\underline{\theta}) = w(\underline{\theta})$. ■

A.2 Proof of Proposition 1

Let $p^*(x, \theta_i)$ be the inverse function of x^* with respect to p. By the first-order condition for the optimal investment, we have

$$p^*(x, \theta_i) = c^{-1}\left(V_x(\theta_i, x)\right) - x.$$

Note that p^* is decreasing in x and

$$\frac{\partial^2 p^*}{\partial x^2}(x,\theta_i) = V_{xxx}(x)\left(c^{-1}\right)'(V_x(\theta_i,x)) + (V_{xx}(\theta_i,x))^2\left(c^{-1}\right)''(V_x(\theta_i,x)).$$

Hence, p^* is convex in x if $V_{xxx} \geq 0$ and $(c^{-1})'' \geq 0$. If p^* is convex in x, x^* is convex in p.

Let P^I and P^{II} be the equilibrium prices in the FPA and SPA, respectively. When x^* is convex in p, we have

$$\begin{aligned} E[x^*(\theta_i, P^{II}) \mid \theta^{(1)} = \theta_i] &> x^*\left(\theta_i, E[P^{II} \mid \theta^{(1)} = \theta_i]\right) \\ &> x^*\left(\theta_i, E[P^{I} \mid \theta^{(1)} = \theta_i]\right) \\ &= x^*\left(\theta_i, P^I\right), \end{aligned}$$

where $\theta^{(1)}$ indicates the highest order statistic of bidders types. Thus, the expected investment is larger in the SPA than in the FPA. ■

A.3 Proof of Proposition 2

From (5) and (6) and by differentiation, we have

$$\frac{\partial(U_{pp}/U_p)}{\partial\theta} = \frac{1}{c^2}\left[\left(x^*_{p\theta}c' + (1+x^*_p)x^*_\theta c''\right)c - (1+x^*_p)x^*_\theta(c')^2\right].$$

We have $x^*_\theta \geq 0$ and $1+x^*_p > 0$ by the single-crossing condition $V_{x\theta} \geq 0$ and concavity of V, respectively. Because C is strictly convex, we have $c, c' > 0$. Hence, U possesses the DARA property if $c'' \leq 0$ and $x^*_{p\theta} \leq 0$.

In addition, remember that

$$x^*_p(\theta_i, p) = \frac{c'(p+x^*)}{V_{xx}(\theta_i, x^*) - c'(p+x^*)}.$$

By differentiation, we have

$$\begin{aligned} x^*_{p\theta} &= \frac{1}{(V_{xx}-c')^2}\left[x^*_\theta c'' \cdot (V_{xx}-c') - c' \cdot \left(x^*_\theta V_{xxx} + V_{xx\theta} - x^*_\theta c''\right)\right] \\ &= \frac{1}{(V_{xx}-c')^2}\left[x^*_\theta V_{xx} c'' - c' \cdot (x^*_\theta V_{xxx} + V_{xx\theta})\right] \end{aligned}$$

When the cost function is quadratic and $C(y) = y + \delta y^2$, we have $C''' = c'' = 0$. Hence, we have $x^*_{p\theta} \leq 0$ if $V_{xxx} \geq 0$ and $V_{xx\theta} \geq 0$. ■

A.4 Proof of Proposition 5

Suppose that there exists a symmetric increasing equilibrium in scoring auctions. Let $\hat{x}^*(\theta_i, s)$ be the optimal investment bid in the first- or second-score auctions, given the bidder's type θ_i and a score s. The winner's optimal investment $\hat{x}^*$ is given by the first-order condition

$$V_x(\theta_i, \hat{x}^*) = \hat{C}_x(\hat{x}^*, s) = \big(1 - q'(\hat{x}^*)\big)\, c\,(s - q(\hat{x}^*) + \hat{x}^*)\,. \tag{12}$$

By differentiation of the implicit function, we have

$$\frac{\partial \hat{x}^*}{\partial s}(\theta_i, s) = -\frac{\hat{C}_{xs}(\hat{x}^*, s)}{\hat{C}_{xx}(\hat{x}^*, s) - V_{xx}(\theta_i, \hat{x}^*)} = -\frac{(1 - q'(\hat{x}^*))c'(s - q(\hat{x}^*) + \hat{x}^*)}{\hat{C}_{xx}(\hat{x}^*, s) - V_{xx}(\theta_i, \hat{x}^*)} < 0,$$

where the inequality holds by the convexity of C and $\hat{C}$ and the concavity of V. Thus, $\hat{x}^*$ is decreasing in s.

Let $\hat{p}^*(\theta_i, s) = s - q\left(\hat{x}^*(\theta_i, s)\right)$ be the price associated with score s and the optimal investment $\hat{x}^*$. Because $\hat{x}^*$ is decreasing in s, $\hat{p}^*$ is increasing in s. Then, let $\hat{p}^{*-1}(\theta_i, p)$ be the inverse function of $\hat{p}^*$ with respect to s. In addition, let $x^{S*}(\theta_i, p) \equiv \hat{x}^*\left(\theta_i, \hat{p}^{*-1}(\theta_i, p)\right)$, which is the optimal investment bid in the first- or second-score auctions, given that the winning price is p. The winner's ex post payoff in terms of their type and a price bid is denoted by

$$U^S(\theta_i, p) \equiv V(\theta_i, x^{S*}(\theta_i, p)) - C(p + x^{S*}(\theta_i, p)).$$

Note that $s - q(\hat{x}^*) + \hat{x}^* = p + \hat{x}^*$. On the one hand, the first-order condition (12) for the optimal investment is written as

$$p + \hat{x}^* = c^{-1}\left(\frac{V_x(\theta_i, \hat{x}^*)}{1 - q'(\hat{x}^*)}\right). \tag{13}$$

On the other hand, the first-order condition for the optimal investment in standard price-only auctions is given by

$$p + x^* = c^{-1}\left(V_x(\theta_i, x^*)\right) < c^{-1}\left(\frac{V_x(\theta_i, x^*)}{1 - q'(x^*)}\right), \tag{14}$$

where the inequality holds by $q' < 1$ by Assumption 3. Equation (14) implies that for any (θ_i, p),

$$x^{S*}(\theta_i, p) = \hat{x}^*\left(\theta_i, \hat{p}^{*-1}(\theta_i, p)\right) > x^*(\theta_i, p). \tag{15}$$

That is, given an auction payment p, the winner invests more in a scoring auction than in a standard auction. Moreover, because

$$U^S(\theta_i, p) \le \max_x V(\theta_i, x) - C(p + x) = U(\theta_i, p),$$

the equilibrium price bid in the first- and second-score auctions must be lower than the equilibrium bid in the FPA and SPA, respectively. It is clear that the equilibrium revenue is lower in scoring auctions than standard auctions. Because the investment function is decreasing in p, the equilibrium ex post investment is greater in scoring auctions than standard auctions by (15). ■

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