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Ryuji Sano
Yokohama National University

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Ryuji Sano†
Department of Economics, Yokohama National University

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Abstract

This study considers revenue-optimal auction design for two ordered substitutes, such as positions or priorities. Bidders have a two-dimensional type about a valuation for the top position and a discount rate of the second position for the top. An auction mechanism is dominant-strategy incentive compatible if and only if it satisfies the "Law of One Price," which requires that bidders' payments are independent of their own discount rate. The simple "virtually efficient mechanism," which maximizes the unconstrained virtual surplus, is not incentive compatible for any type distribution if the discount-rate-type space includes at least two interior values. If the discount-rate-type space includes at most a single interior value, there exist distributions under which the virtually efficient mechanism is incentive compatible, and therefore optimal.

Keywords: multi-object auction, revenue maximization, multidimensional type
JEL Classification Codes: D82, D44

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†Department of Economics, Yokohama National University, Tokiwadai 79-4, Hodogaya, Yokohama 240-8501, Japan. Telephone: +81-45-3393563. E-mail: sano-ryuji-cx@ynu.ac.jp

1 Introduction

This study considers optimal auctions of priorities, such as service slots for facilities and positions of Internet ads. Consider a queueing problem in which a seller owns a single facility and allocates the service provision time to many potential users. The users want to be served as soon as possible. A late slot is acceptable, however, users are not patient and discount the value of delayed service. The degree of patience is heterogeneous between users and their private information. When users have a two-dimensional type, a valuation of the service and a patience level, how should the seller design an auction to maximize their expected profits?[1]

When bidders have a single-dimensional type regarding valuations, the optimal single-object auction for the seller is fully characterized by Myerson (1981). By the envelope condition, the expected payment from a bidder is expressed in terms of their *virtual valuation*. When the type distribution satisfies certain regularity conditions, then the allocation rule that maximizes the social surplus in terms of virtual valuation – the *virtually efficient allocation rule* – is incentive compatible and optimal for the seller. Under reasonable assumptions, the optimal allocation rule is implemented by a standard auction, with an appropriate reserve price. Myerson's virtual valuation approach is applicable to multiple objects as long as the bidders' type is single-dimensional (Monteiro, 2002; Ulku, 2013).

However, it is well known that it is hard to obtain an optimal mechanism when bidders have multidimensional types. Myerson's approach of using virtual valuation is not successful very much because the multidimensional extension of the virtual valuation is not unique but is endogenously defined by allocation rules. Even for screening problems, where there is only one buyer, the optimal sales mechanism is in general stochastic and highly complex (Thanassoulis, 2004; Manelli and Vincent, 2007; Pavlov, 2011; Daskalakis et al. 2017; etc.). The existing studies imply that optimal auctions for multidimensional types will be extremely complex.[2]

[1] We use they for single pronoun.

[2] Chen et al. (2019) show that every Bayesian incentive compatible allocation rule is implemented by a deterministic mechanism when there are multiple agents. However, the construction of such a deterministic mechanism is not easy or intuitive.

In contrast, there are some multidimensional type models in which Myerson's approach can be applied. For example, Iyengar and Kumar (2008), Malakhov and Vohra (2009), and Devanur et al. (2020) examine auctions or sales mechanisms with capacity-constrained agents. In these models, each agent has a two-dimensional type about a marginal valuation of the good and a capacity (maximum demand).[3] Myersonian virtual valuation is constructed by the incentive condition on the valuation type, and there exist type distributions where the virtually efficient allocation rule is incentive compatible and optimal. Related models include Dizdar et al. (2011) (multi-unit demands with perfect complements), Pai and Vohra (2013), Mierendorff (2016), and Sano (2021) (dynamic allocations with private participation time).

In these models, each bidder's type is classified into two components, a single-dimensional valuation, and other attributes which specify the conditions for obtaining the valuation. Sufficient conditions for the incentive compatibility of the virtually efficient mechanism are different. They include hazard rate ordering (Iyengar and Kumar, 2008; Malakhov and Vohra, 2009; Dizdar et al., 2011), convex virtual valuations (Mierendorff, 2016), and affine virtual valuations (Sano, 2021).

This study aims to explore an intermediate between these multidimensional type models. We focus on a simultaneous auction of two ordered substitutes, which are priorities or positions. Each bidder has a unit-demand and a valuation v for the top priority or the higher position, which we call the *Top*. The valuation for the second priority or the lower position, which we call the *Bottom*, is determined by δv, where $\delta \in [0, 1]$ is the discount rate of the Bottom for the Top. A pair (v, δ) is the private information of a bidder. In the queueing problem, v is a valuation of the service, and δ represents a level of patience or waiting cost. In position auctions such as Google's sponsored search, v is the expected profit from a click on an ad link, and δ represents the position-specific attractiveness (or click-through rate) of the ad. If the set of possible discount rates, denoted by $\Delta \subseteq [0, 1]$, is binary and $\Delta = \{0, 1\}$, our model is technically similar to the existing "value and other attributes" models. The richer the discount rates set Δ, the closer the model will be to a general model

[3]Precisely, Iyengar and Kumar (2008) study multi-unit procurement auctions with capacity-constrained suppliers.

with various valuations over different outcomes.

We characterize situations in which the virtually efficient mechanism is dominant-strategy incentive compatible (DSIC) and maximizes the seller's expected revenue. Although DSIC is stronger than Bayesian incentive compatibility, which is standard in the literature on optimal mechanism design, we will have a clear characterization using DSIC.

We show that when there are two or more interior discount rates in Δ, the virtually efficient mechanism is not DSIC for any type distribution. This negative result is shown as follows. The so-called taxation principle implies that DSIC of an auction mechanism is characterized by the "Law of One Price (LOP)". The condition requires that, given the other bidders' types, each bidder's payment for a position is independent of their own valuation or discount rate. The efficient allocation rule, which is implemented by the Vickrey-Clarke-Groves (VCG) mechanism, should satisfy the LOP condition. Therefore, the virtually efficient allocation rule also should satisfy the LOP in terms of virtual valuations. However, the LOP as the incentive condition and the LOP in terms of virtual valuations are not compatible, except for the cases where the virtual valuation takes special forms.

Possibility results arise when the domain of the possible discount rates is $\Delta \subseteq \{0, \delta, 1\}$ with $0 < \delta < 1$. This is because LOP is not necessarily required for the corner discount rates of 0 and 1. Specifically, when the set of discount rates is $\Delta = \{0, 1\}$, the virtually efficient mechanism is DSIC and optimal if the type distribution satisfies the hazard rate order. This implies that the virtual valuation is monotone in discount rate. This is a replication of the results from Malakhov and Vohra (2009), Dizdar et al. (2011), and Pai and Vohra (2013). When Δ includes an interior value δ, the condition for DSIC of the virtually efficient mechanism is more restrictive. When $\Delta = \{0, \delta\}$ and valuation and discount rate are independently distributed, the virtually efficient mechanism is DSIC if the virtual valuation function is affine. A similar result is given by Sano (2021) in a dynamic auction model. When $\Delta = \{\delta, 1\}$ or $\{0, \delta, 1\}$, the virtually efficient mechanism is DSIC only in knife-edge type distributions. All these results confirm that it is difficult to extend Myerson's approach to multidimensional types, even if the cardinality of the discount-rate type is very small.

The generic impossibility of DSIC of the virtually efficient mechanism is partly due to the strictness of dominant strategy. We also show that among a class of symmetric mechanisms, the efficient (VCG) mechanism is a unique mechanism that satisfies LOP and therefore DSIC. The literature of mechanism design with fairness concerns shows that the efficiency is characterized by DSIC and the symmetry of an allocation rule under single-dimensional type (Ashlagi and Serizawa, 2012; Hashimoto and Saito, 2012). Our result can be viewed as an extension to a multidimensional restricted type space.

1.1 Related Literature

General models of multidimensional mechanism design are studied by Manelli and Vincent (2007) and Daskalakis et al. (2017), among others. For allocation of the perfect substitutes, Thanassoulis (2004) and Pavlov (2011) show that in the case of single agent the optimal mechanism is generally stochastic.

There are a number of special multidimensional type models with valuations and other attributes. Malakhov and Vohra (2009), Devanur et al. (2020), and Iyengar and Kumar (2008) study capacity-constrained agents. Malakhov and Vohra (2009) and Devanur et al. (2020) consider allocations of homogeneous goods to an agent with maximum consumption units. Iyengar and Kumar (2008) examine multi-unit procurement auctions with capacity-constrained suppliers. Malakhov and Vohra (2009) and Iyengar and Kumar (2008) provide hazard rate ordering as a sufficient condition for the incentive compatibility of the relaxed solution. Devanur et al. (2020) show that the optimal allocation rule is deterministic even if the relaxed solution is not incentive compatible. Dizdar et al. (2011) consider the allocations of homogeneous goods to agents with multi-unit demands. In their model, agents demand multiple units as perfect complements, and hazard rate ordering is a sufficient condition for the incentive compatibility of the relaxed solution. Pai and Vohra (2013) and Mierendorff (2016) consider dynamic sales models with private consumption deadlines. Pai and Vohra (2013) gives hazard rate ordering as a sufficient condition, whereas convex virtual valuation is proposed in Mierendorff (2016). Sano (2021) characterizes affine virtual valuations as the incentive compatiblity of the relaxed solution in a related dynamic model.

Mishra and Roy (2013) consider a more general dichotomous preference model, in which agents have the (same) valuation if a preferable outcome is chosen. The set of preferable outcomes to each agent is their private information. They show that if the valuation and other private information are independently distributed, the virtually efficient mechanism is DSIC and maximizes the expected revenue.

Our study is also related to multi-dimensional mechanism design that seeks models in which a simple mechanism is optimal. In single-agent multi-product monopoly models, Manelli and Vincent (2006) and Haghpanah and Hartline (2020) provide conditions under which pure bundling is optimal. In contrast, Carroll (2017) shows that no bundling can be optimal if the seller knows the buyer's marginal distributions only and has ambiguity-averse preferences.

For the strictness of DSIC, Roberts (1979) shows that DSIC implies that allocation rule must be affine maximizer under the unrestricted type space. For the restricted domains such as auctions, Ashlagi and Serizawa (2012) shows DSIC and anonymity in utility induces efficiency in multi-unit auctions with unit-demand bidders. Hashimoto and Saito (2012) show a similar result for a queueing problem. Mishra and Quadir (2014) show that DSIC and non-bossiness with a technical condition regarding continuity induces utility maximizer of allocation rules in single-object auctions. For multidimensional types, Kazumura et al. (2020) show that when there are at least three goods and bidders have unit-demand, DSIC and anonymity (and individual rationality and no wastage) do not induce efficiency. They characterize the VCG (formally, the minimum Walrasian equilibrium mechanism) using ex-post revenue maximization. Our model is distinct from these results in that the type space is more restricted and non-convex, but we explicitly assume utility maximizing allocations.

2 Model

Suppose that the seller allocates two positions T (the Top, a superior good) and B (the Bottom, a normal good) to many buyers. There are I (≥ 3) potential buyers, who each have a single-unit demand. The set of potential buyers is denoted by

$I = \{1, \ldots, |I|\}$. An allocation for bidder i is denoted by $x_i \in \{T, B, 0\}$, where 0 represents a null good. An allocation is denoted by $x = (x_i)_{i \in I}$. An allocation x is feasible if for all $i \in I$, it holds that $x_i \in \{T, B\} \Rightarrow x_j \neq x_i \ (\forall j \neq i)$. The set of all feasible allocations is denoted by X.

Bidders have quasi-linear utility. Bidder i's type is denoted by $\theta_i = (v_i, \delta_i)$, where $v_i \in \mathbb{R}_+$ is the valuation of the top T and $\delta_i \in [0, 1]$ is the discount rate of the bottom B for the top. Given a monetary transfer $p_i \in \mathbb{R}$ to the seller, bidder i's utility function takes the form

$$u_i = \begin{cases} v_i - p_i & \text{if } x_i = T \\ \delta_i v_i - p_i & \text{if } x_i = B \\ -p_i & \text{if } x_i = 0 \end{cases} \cdot \tag{1}$$

The set of all types for a bidder is denoted by $\Theta \equiv V \times \Delta$. We assume that the set of valuations is unbounded and $V = \mathbb{R}_+$. The set of discount rates $\Delta \subset [0, 1]$ satisfies $|\Delta| \geq 2$. Let $\bar{\delta} \equiv \max_{\delta \in \Delta} \delta$ be the maximum possible discount rate. A type profile of the bidders is denoted by $\theta = (\theta_i)_{i \in I}$.

We focus on deterministic direct mechanisms. A mechanism is denoted by (x, p), in which $x : \Theta^I \to X$ is an allocation rule and $p : \Theta^I \to \mathbb{R}^I$ is a payment rule. We assume that the bottom B is not allocated to bidders with no value as follows.

Assumption 1 The bottom B is not allocated to bidder i if their reported discount rate is $\delta_i = 0$.

Bidder i's report in a mechanism is denoted by $\hat{\theta}_i = (\hat{v}_i, \hat{\delta}_i)$. Given a mechanism (x, p) and a report profile $\hat{\theta} = (\hat{\theta}_i)_{i \in I}$, bidder i's payoff function is given by

$$u_i(\hat{\theta}, \theta_i) = \chi_i(\hat{\theta}; \delta_i) v_i - p_i(\hat{\theta}), \tag{2}$$

where

$$\chi_i(\hat{\theta}; \delta_i) = \begin{cases} 1 & \text{if } x_i(\hat{\theta}) = T \\ \delta_i & \text{if } x_i(\hat{\theta}) = B \\ 0 & \text{if } x_i(\hat{\theta}) = 0 \end{cases} \cdot \tag{3}$$

Truthful payoff is denoted by

$$U_i(\theta) \equiv u_i(\theta, \theta_i) = \chi_i(\theta; \delta_i)v_i - p_i(\theta). \tag{4}$$

We consider DSIC as the equilibrium concept.

Definition 1 A mechanism (x, p) is *dominant-strategy incentive compatible (DSIC)* if for all $i \in I$, all $\theta \in \Theta^I$, and all $\hat{\theta}_i \in \Theta$,

$$U_i(\theta) \geq u_i((\hat{\theta}_i, \theta_{-i}), \theta_i).$$

Definition 2 A mechanism (x, p) is *individually rational (IR)* if for all $i \in I$ and all $\theta \in \Theta^I$, $U_i(\theta) \geq 0$.

Our main objective is to find a DSIC mechanism that maximizes the seller's expected revenue.

$$\max_{(x,p)|x \in X} E\left[\sum_{i \in N} p_i(\theta)\right]$$
$$\text{s.t.} \quad DSIC, IR$$

Although most studies on optimal mechanism design consider Bayesian incentive compatible mechanisms, we focus on DSIC mechanisms in this study. We will have a clear result by imposing a stronger incentive condition. Ulku (2013), Mishra and Roy (2013), and Kazumura et al. (2020) also examine revenue maximization among DSIC mechanisms. Regarding deterministic mechanisms, Chen et al. (2019) show that for any probabilistic Bayesian incentive compatible mechanism, there exists an equivalent deterministic Bayesian incentive compatible mechanism.

3 Characterization of DSIC

Because we consider DSIC as the equilibrium concept, we fix an arbitrary θ_{-i} and omit from description. We consider as if there is only one bidder in the mechanism. In addition, we abuse notations and denote by $\chi_i(v_i, \delta_i)$ the case where the true discount rate is reported, $\chi_i(v_i, \delta_i) = \chi_i(v_i, \delta_i; \delta_i)$.

If the discount rate δ_i is known to the seller, the incentive compatibility (in valuation) is characterized in a standard manner by the monotonicity and the envelope conditions.

Definition 3 An allocation rule x is *monotone* if for each $\delta_i \in \Delta$,

$$v_i > v_i' \Rightarrow \chi_i(v_i, \delta_i) \geq \chi_i(v_i', \delta_i).$$

A mechanism (x, p) is said to be monotone if x is monotone.

Lemma 1 *If a mechanism is DSIC, then it is monotone.*

Proof. Standard and omitted.

Because we focus on deterministic mechanisms, a monotone allocation rule is characterized by *cutoff functions*. Given a monotone allocation rule x, the cutoff for allocation $x_i \in \{T, B\}$ is defined as follows.[4]

$$c_i^T(\delta_i) \equiv \inf\{v_i \mid \chi_i(v_i, \delta_i) = 1\} \tag{5}$$

$$c_i^B(\delta_i) \equiv \inf\{v_i \mid \chi_i(v_i, \delta_i) \geq \delta_i\} \quad (\delta_i > 0) \tag{6}$$

For the completeness, let $c_i^B(0) \equiv c_i^T(0)$. Also, note that $c_i^B(1) = c_i^T(1)$ by definition. Let $c_i^x(\delta_i) \equiv \infty$ for $i \in I$ and $x \in \{T, B\}$ if the infimum does not exist. When $0 < c_i^B(\delta_i) < c_i^T(\delta_i) < \infty$, the allocation rule is given by

$$x_i(\theta_i) = \begin{cases} T & \text{if } v_i > c_i^T(\delta_i) \\ B & \text{if } c_i^B(\delta_i) < v_i < c_i^T(\delta_i) \\ 0 & \text{if } v_i < c_i^B(\delta_i) \end{cases},$$

while allocations at the cutoffs are not specified.

The envelope condition in valuation is stated as follows. Note that type $\theta_i = (0, \delta_i)$ with any δ_i represents the same preferences. Thus, the truthful payoff must satisfy $U_i(0, \delta_i) = \underline{U}_i$ for all $\delta_i \in \Delta$.

Lemma 2 *If a mechanism is DSIC, then for all $\theta_i \in \Theta$, the truthful payoff satisfies*

$$U_i(\theta_i) = \underline{U}_i + \int_0^{v_i} \chi_i(s, \delta_i) \mathrm{d}s. \tag{7}$$

[4]Note that the cutoffs actually depend on θ_{-i} and should be denoted by $c_i^{x_i}(\delta_i, \theta_{-i})$.

Proof. Standard and omitted.

The payment rule is pinned down by the envelope condition up to constant. When the bidder has a type θ_i satisfying $c_i^B(\delta_i) < v_i < c_i^T(\delta_i)$, then they obtain the bottom B and pay

$$p_i(\theta_i) = p_i^B(\delta_i) \equiv -\underline{U}_i + \delta_i c_i^B(\delta_i). \tag{8}$$

If $v_i > c_i^T(\delta_i)$, then they obtain the top T and pay

$$p_i(\theta_i) = p_i^T(\delta_i) \equiv -\underline{U}_i + (1-\delta_i)c_i^T(\delta_i) + \delta_i c_i^B(\delta_i). \tag{9}$$

Note that an allocation rule may not assign a specific position to the bidder. When the bottom B is not allocated to the bidder of δ_i for any v_i, then $c_i^B(\delta_i) = c_i^T(\delta_i)$. The following lemma shows that all cutoffs are finite if the top T is allocated to some type θ_i.

Lemma 3 *Suppose that for all $\theta_{-i} \in \Theta^{I-1}$, there exists $\theta_i \in \Theta$ and $x_i(\theta_i, \theta_{-i}) = T$. If a mechanism is DSIC, then $c_i^B(\delta_i) \leq c_i^T(\delta_i) < \infty$ for all $\delta_i \in \Delta$ and all $\theta_{-i} \in \Theta^{I-1}$.*

Proof. See Appendix.

The incentive compatibility with respect to discount rates is given by the so-called taxation principle. Consider two discount rates $\delta_i, \delta_i' > 0$, and suppose $c_i^B(\delta_i) < c_i^T(\delta_i)$ and $c_i^B(\delta_i') < c_i^T(\delta_i')$. Both positions are priced according to (8) and (9) for each discount rate. It is clear that if the prices are different between the discount rates, no bidder has an incentive to purchase at a higher price, and the mechanism is not DSIC. Hence, DSIC requires the equivalent payments between discount rates.

The taxation principle implies that if a mechanism is DSIC, then for each θ_{-i}, there exists a price vector $p_i = (p_i^B, p_i^T)$, and bidder i purchases the best position among $\{T, B, 0\}$ under p_i. Given an arbitrary θ_{-i} and a monotone allocation rule with cutoff functions $(c_i^B, c_i^T)_{\delta_i \in \Delta}$, the taxation principle implies the *Law of One Price (LOP)*, which requires $p_i^{x_i}(\delta_i) = p_i^{x_i}$ for each $x_i \in \{T, B\}$ and most of $\delta_i \in \Delta$. Lemma 3 indicates that LOP for the top T holds for all $\delta_i \in \Delta \setminus \{1\}$, however, it may not for $\delta_i = 1$ because the bidder enjoys v_i from the bottom B, and the top T may not be allocated. In addition, LOP for the bottom B is not required for small δ_i because the bottom is neither very valuable nor the bidder wants to purchase at p_i^B. Hence, DSIC is formally characterized using cutoffs as follows.

Proposition 1 *A monotone mechanism with a profile of cutoffs* $\left(c_i^B(\delta), c_i^T(\delta)\right)_{\delta\in\Delta}$ *is DSIC for bidder* i *if and only if there exists a price vector* $(p_i^B, p_i^T) \in \mathbb{R}_+^2$ *and it satisfies* $p_i^B \leq p_i^T$ *and the following properties:*

1. *for all* $\delta < 1$,
$$(1-\delta)c_i^T(\delta) + \delta c_i^B(\delta) = p_i^T, \tag{LOP-T}$$

2. *for all* $\delta \geq p_i^B/p_i^T$,
$$\delta c_i^B(\delta) = p_i^B, \tag{LOP-B}$$

3. *if* $\bar{\delta} = 1$ *and* $p_i^B < p_i^T$*, then* $x_i(v_i, 1) \in \{0, B\}$ *for all* v_i*, and*

4. *if* $\delta < p_i^B/p_i^T$*, then* $x_i(v_i, \delta) \in \{0, T\}$ *for all* v_i*.*

The third and last requirements are complementary conditions to the first and second, respectively. We call these conditions the LOP conditions or simply LOP. Note that LOP does not indicate one price for different bidders. Also, prices for bidder i depends on the other bidders' types θ_{-i}.

4 Main Result

Suppose that each bidder's type θ_i is independently and identically distributed. Denote by G the cumulative distribution function of θ_i. Given a discount rate δ_i, the conditional hazard rate is denoted by

$$\lambda_{\delta_i}(v_i) \equiv \frac{g(v_i \mid \delta_i)}{1 - G(v_i \mid \delta_i)}. \tag{10}$$

We impose increasing hazard rate, which is often assumed in auction literature.

Assumption 2 Conditional hazard rate λ_{δ_i} is strictly increasing in v_i for all $\delta_i \in \Delta$.

The *virtual valuation* is defined as

$$\phi(\theta_i) \equiv v_i - \frac{1}{\lambda_{\delta_i}(v_i)}. \tag{11}$$

When v_i and δ_i are independently distributed, we simply denote $\lambda_{\delta_i}(v_i) = \lambda(v_i)$ and $\phi(\theta_i) = \phi(v_i)$. By standard calculations, the expected revenue from bidder i is transformed into the expected virtual surplus:

$$E[p_i(\theta)] = E[\chi_i(\theta)\phi(\theta_i)]. \tag{12}$$

The *pointwise relaxed problem* is the unconstrained virtual surplus maximization problem.

$$\max_{x(\theta)\in X} \sum_{i\in I} \chi_i(\theta)\phi(\theta_i). \tag{13}$$

The solution to (13) for $\theta \in \Theta^I$ is called the *relaxed solution* for θ. If the relaxed solution satisfies the LOP conditions in Proposition 1 for all θ, then it is the optimal allocation rule that maximizes the seller's expected revenue. In addition, it is clear that the optimal allocation rule maximizes the seller's expected revenue among all Bayesian incentive compatible mechanisms.

Suppose $\{\delta_l, \delta_h\} \subset \Delta$ with $0 < \delta_l < \delta_h < 1$. For a while, suppose that the valuation v_i and the discount rate δ_i are independently distributed. Because the efficient allocation rule is implementable, it satisfies the LOP conditions. When $c_i^B(\delta_l) < c_i^T(\delta_l)$, the LOP conditions (LOP-T) and (LOP-B) require

$$\delta_h c_i^B(\delta_h) = \delta_l c_i^B(\delta_l), \tag{14}$$

$$(1-\delta_h)c_i^T(\delta_h) = (1-\delta_l)c_i^T(\delta_l). \tag{15}$$

Now let us turn to the relaxed problem. The relaxed problem is the same as social surplus maximization, except that the valuations are replaced with the virtual valuations. Hence, the relaxed solution must satisfy the LOP conditions in terms of virtual valuations, which are

$$\delta_h \phi\left(c_i^B(\delta_h)\right) = \delta_l \phi\left(c_i^B(\delta_l)\right), \tag{16}$$

$$(1-\delta_h)\phi\left(c_i^T(\delta_h)\right) = (1-\delta_l)\phi\left(c_i^T(\delta_l)\right). \tag{17}$$

The relaxed solution is DSIC if the LOP and the "virtual" LOP coincide. However, it holds only if the virtual valuation is linear: $\phi(v) = \alpha v$, which never holds because $\phi(0) < 0$. Hence, the relaxed solution is not DSIC for any value distribution. This is similar when v_i and δ_i are correlated. The following theorem is our main result.

Theorem 1 *Suppose* $\Delta \supseteq \{\delta_l, \delta_h\}$ *with* $0 < \delta_l < \delta_h < 1$. *The relaxed solution is not DSIC for any type distribution* G.

Thus, possibility results arise only when there is at most a single discount rate in the interior of the unit interval $(0, 1)$; that is, $\Delta \subseteq \{0, \delta, 1\}$. This is because LOP conditions are partly slack for $\delta_i = 0$ and 1. When discount rate is $\delta_i = 0$, the LOP for the bottom B is not required. When discount rate is $\delta_i = 1$, the LOP for the top T is not required.

4.1 Possibility 1: $\Delta = \{0, \delta\}$

To have a possibility result, let us suppose $\Delta = \{0, \delta\}$ with $\delta \in (0, 1)$ first. Then, Proposition 1 is reduced as follows.

Corollary 1 *Suppose* $\Delta = \{0, \delta\}$ *with* $\delta \in (0, 1)$. *A monotone mechanism with cutoffs* $(c_i^B(\delta), c_i^T(\delta), c_i^T(0))$ *is DSIC for bidder i if and only if*

$$(1 - \delta)c_i^T(\delta) + \delta c_i^B(\delta) = c_i^T(0). \tag{18}$$

Equation (18) corresponds to (LOP-T). (LOP-B) is not required because bidders of $\delta_i = 0$ are not allocated the bottom. Because the efficient allocation rule satisfies (18), the relaxed solution satisfies the associated virtual LOP condition

$$(1 - \delta)\phi_\delta(c_i^T(\delta)) + \delta\phi_\delta(c_i^B(\delta)) = \phi_0(c_i^T(0)). \tag{19}$$

It is clear that given that v_i and δ_i are independent, two conditions (18) and (19) coincide if the virtual valuation ϕ is affine. Hence, we have the following sufficient condition for DSIC of the relaxed solution in the case $\Delta = \{0, \delta\}$.

Theorem 2 *Suppose* $\Delta = \{0, \delta\}$ *with* $\delta \in (0, 1)$ *and that* v_i *and* δ_i *are independently distributed. The relaxed solution is DSIC and therefore optimal if the virtual valuation function* ϕ *is affine. Conversely, if the virtual valuation is not affine, then there exists a type profile* θ_{-i} *such that the relaxed solution does not satisfy the LOP conditions for bidder i.*

Virtual valuation function is affine when the value distribution is uniform or exponential. However, generically, virtual valuation is not affine, or the relaxed solution is not DSIC even if v_i and δ_i are independently distributed.

4.2 Possibility 2: $\Delta = \{\delta, 1\}$

Suppose $\Delta = \{\delta, 1\}$ with $\delta \in (0,1)$. Then, the characterization of DSIC is reduced as follows.

Corollary 2 *Suppose $\Delta = \{\delta, 1\}$ with $\delta \in (0,1)$. A monotone mechanism with cutoffs $(c_i^B(\delta), c_i^T(\delta), c_i^B(1))$ is DSIC for bidder i if and only if for each θ_{-i}, either one of the following conditions hold:*

1. *$c_i^B(\delta) < c_i^T(\delta)$ and*

$$\delta c_i^B(\delta) = c_i^B(1), \tag{20}$$

or

2. *$c_i^B(\delta) = c_i^T(\delta)$ and*

$$\delta c_i^B(\delta) \leq c_i^B(1) \leq c_i^B(\delta). \tag{21}$$

In the current specification of Δ, (LOP-T) is not necessary because bidders of $\delta_i = 1$ may not want to purchase the top. The former condition of the corollary corresponds to (LOP-B). The latter corresponds to the case in which bidders of $\delta_i = \delta$ are not allocated the bottom. The bottom is allocated only to bidders of $\delta_i = 1$.

The most restrictive condition in the corollary is (20). The associated condition in terms of virtual valuations,

$$\delta \phi_\delta(c_i^B(\delta)) = \phi_1(c_i^B(1)),$$

satisfies (20) if the virtual valuation satisfies for all v_i,

$$\delta \phi_\delta(v_i) = \phi_1(\delta v_i).$$

This is possible in very specific type distributions as follows.

Theorem 3 *Suppose $\Delta = \{\delta, 1\}$ with $0 < \delta < 1$. The relaxed solution is DSIC and therefore optimal if the type distribution satisfies for all $v_i \geq r_\delta \equiv \phi_\delta^{-1}(0)$,*

$$G(v_i \mid \delta) = G(\delta v_i \mid 1). \tag{22}$$

An example of distribution G satisfying (22) is exponential distributions. When $G(\cdot|1)$ is an exponential distribution with hazard rate λ, $G(\cdot|\delta)$ is also an exponential distribution with hazard rate $\delta\lambda$. Note that (22) implies that the value distribution conditional on δ stochastically dominates that on 1 in terms of hazard rate.

4.3 Other Possibilities

Suppose $\Delta = \{0, 1\}$. Proposition 1 is reduced as follows.

Corollary 3 *Suppose $\Delta = \{0, 1\}$. A monotone mechanism with cutoffs $(c_i^T(0), c_i^B(1))$ is DSIC for bidder i if and only if for all $\theta_{-i} \in \Theta^{I-1}$, $c_i^T(0) \geq c_i^B(1)$.*

The LOP is not required in this case. Because the bottom is allocated only to those of discount rate $\delta_i = 1$, the LOP for the bottom is not necessary. Although the top may be allocated to both discount types, the LOP for the top is dropped by the following consideration. When a bidder of $\delta_i = 1$ obtains either slot, they pay $c_i^B(1)$. When a bidder of $\delta_i = 0$ obtains the top, they pay $c_i^T(0)$. Suppose $c_i^T(0) > c_i^B(1)$, which means that the slots are price discriminated by discount rates. A bidder of type $(v_i, 0)$ with $v_i > c_i^B(1)$ may have an incentive to pretend to be $\delta_i = 1$, but it is not profitable. This is because when slots are price discriminated by discount rates, the bidder of $\delta_i = 1$ is allocated the bottom regardless of the reported valuation type $\hat{v}_i > c_i^B(1)$.

Because DSIC is characterized by an inequality, the sufficient condition for the relaxed solution to be DSIC is relatively weak.

Theorem 4 *Suppose $\Delta = \{0, 1\}$. The relaxed solution is DSIC and therefore optimal if $\lambda_1(v_i) \geq \lambda_0(v_i)$ for all v_i; that is, the valuation-type distribution conditional on $\delta_i = 0$ weakly dominates that conditional on $\delta_i = 1$ in terms of hazard rate.*

The hazard rate ordering implies that the virtual valuation is monotone in discount rate. The case $\Delta \equiv \{0, 1\}$ is technically similar to Malakhov and Vohra (2009), Iyengar and Kumar (2008), and Dizdar et al. (2011), all of which give the same sufficient condition for the incentive compatibility.

Finally, the case of the maximal domain of Δ is stated as follows. If the type distribution satisfies all the conditions imposed so far, the relaxed solution is DSIC.

Theorem 5 *Suppose $\Delta = \{0, \delta, 1\}$ with $\delta \in (0, 1)$. The relaxed solution is DSIC and therefore optimal if for all v_i,*

$$G(v_i \mid 0) = G(v_i \mid \delta) = G(\delta v_i \mid 1) \tag{23}$$

and $\phi(v_i, \delta)$ is affine in v_i.

4.4 Efficiency

When the relaxed solution is DSIC, the optimal allocation rule is implemented by the Vickrey-Clarke-Groves mechanism in terms of virtual valuation. This does not imply that the optimal allocation rule is efficient in terms of actual valuation. The social welfare is

$$v_i + \delta_j v_j,$$

where bidders i and j are those who are assigned the Top T and the Bottom B, respectively. The virtual surplus, which is maximized in the relaxed problem, is

$$\phi(v_i, \delta_i) + \delta_j \phi(v_j, \delta_j).$$

Clearly, the relaxed solution is not efficient if the valuation and discount-rate types are correlated and $\phi(\cdot, \delta_i) \neq \phi(\cdot, \delta_i')$.

When the valuation and discount-rate types are independently distributed, the optimal mechanism is efficient (for positive virtual valuations) if $\Delta = \{0, \delta\}$ with $\delta \in (0, 1]$ and the virtual valuation is affine. Under the specification, the virtual surplus is

$$\phi(v_i) + \delta_j \phi(v_j) = \alpha(v_i + \delta_j v_j) + \beta(1 + \delta_j),$$

where $\phi(v_i) = \alpha v_i + \beta$. Note that the Bottom can be allocated to only bidders with $\delta_j = \delta > 0$ by assumption. Hence, we have $\beta(1+\delta_j) = \beta(1+\delta)$, which is constant as long as both slots are allocated. By introducing a reservation type $\theta_0 = (r, \delta)$ with $r = -\beta/\alpha$, the slots are apparently always allocated. Thus, the optimal mechanism is the Vickrey-Clarke-Groves mechanism with a reserve price r.

Proposition 2 *The optimal mechanism is efficient for positive virtual valuations if $\Delta = \{0, \delta\}$ with $\delta \leq 1$, valuation and discount-rate types are indepnedently dis-*

tributed, and the virtual valuation ϕ is affine. Under the specification, the Vickery-Clarke-Groves mechanism with a reserve price $r = \phi^{-1}(0)$ is optimal.

5 Symmetry and Efficiency

As we have seen in the previous section, the LOP in Proposition 1 is so restrictive that the relaxed solution hardly satisfies it. Moreover, the following example shows that even a simple assortative allocation rule is not implementable.

Example 1 (The "assortative allocation" is not implementable.) Consider the allocation rule in which goods are allocated in an assortative manner as follows. The seller allocates the top T to the highest bidder and the bottom B to the bidder with the highest valuation of $\delta_i v_i$ among the others. This allocation rule is not implementable. Specifically, suppose $I = \{1, 2, 3\}$, $\Delta \supseteq \{\delta_l, \delta_h\}$ with $0 \leq \delta_l < \delta_h \leq 1$, $\theta_2 = (v_2, \delta_l)$, and $\theta_3 = (v_3, \delta_l)$ with $v_3 < v_2$. The allocation rule implies that the cutoff value of bidder 1 for the top is $c_1^T(\delta_1) = v_2$ for every discount rate $\delta_1 \in \Delta$. When $\delta_1 \neq 0$ and $v_1 \in (\frac{\delta_l}{\delta_1} v_3, v_2)$, bidder 1 is allocated the bottom and $c_1^B(\delta_1) = \frac{\delta_l}{\delta_1} v_3$ for all $\delta_1 \neq 0$.

Bidder 1's payoff is $\delta_1 v_1 - \delta_l v_3$ for $\delta_1 \in \Delta \setminus \{0\}$ if the bottom is allocated. Hence, bidder 1's net willingness to pay for the top is $v_1 - \delta_1 v_1 + \delta_l v_3$. Given the cutoff $c_1^T(\delta_1) = v_2$, the incentive compatibility requires $p_1^T = (1-\delta_1)v_2 + \delta_l v_3$, which depends on δ_1 and violates LOP. Thus, this allocation rule is not implementable.

Note that the assortative allocation rule above is not efficient because the bidder of the highest v_i should be assigned the bottom if they have a large discount rate. The efficient allocation is given as follows. Let $v_1 > v_2 > \cdots > v_n$. The efficient allocation chooses the better allocation between

- The top T is allocated to bidder 1. The bottom B is allocated to bidder j ($\neq 1$) satisfying $\delta_j v_j = \max_{k \neq 1} \delta_k v_k$, and
- The top T and the bottom B are allocated to bidders 2 and 1, respectively.

In the rest of this section, we show that the efficient allocation rule is a unique, implementable rule among a reasonable class of symmetric mechanisms.

Suppose that an allocation rule maximizes a "utility function"

$$F(v^T, v^B, \delta^B), \tag{24}$$

where (v^T, δ^T) and (v^B, δ^B) are types to whom positions T and B are allocated, respectively. The allocation rule is clearly symmetric between bidders. We additionally impose the following conditions.

Assumption 3 The utility function F satisfies the following properties.

1. F is continuous and strictly increasing in v^T, v^B, and δ^B.
2. Suppose $\delta < 1$. Then, $F(v, v', \delta) > F(v', v, \delta)$ if and only if $v > v'$.

In addition to these assumptions, we are implicitly or explicitly imposing the following assumptions:

- Both positions are necessarily allocated to bidders (generically).
- The utility maximizer implies non-bossiness of the allocation rule (Mishra and Quadir, 2014).
- The utility function does not depend on δ^T, whereas the virtual surplus does via the virtual valuation function $\phi(\cdot, \delta_i)$.

The main result of this section is as follows.

Theorem 6 *Suppose $\Delta \subseteq (0, 1)$. An allocation rule x is implementable in weakly dominant strategy and maximizes a utility function F, which satisfies Assumption 3, if and only if x is efficient.*

Theorem 6 implies LOP or DSIC is much restrictive in symmetric mechanism design. In the literature on mechanism design with fairness concern, Ashlagi and Serizawa (2012) and Hashimoto and Saito (2012) show that in single-dimensional type models, the efficient allocation rule is induced by DSIC and symmetry. For multi-dimensional type model, Kazumura et al. (2020) show that when there are at least three goods and bidders have unit-demand, DSIC and symmetry (and other natural axioms) do not induce efficiency. In our model, efficiency is induced by symmetric utility maximizer under restricted type space. It would be worth noting that efficiency is induced even if the discount-rate-type is just binary.

6 Conclusion

We have examined the conditions under which the virtually efficient mechanism is DSIC and maximizes the seller's expected revenue in a multi-object auction with multidimensional types. When bidders have two-dimensional type about the valuation for the superior good and the discount rate of the inferior good, the Myersonian virtually efficient mechanism cannot be DSIC except for some special specifications on the domains and distributions of type. Our result confirms that it is hard to extend Myerson's (1981) approach to optimal auction design with multidimensional type.

The negative result is due to the strictness of weakly dominant strategy. We also showed that only the efficient mechanism is implementable in weakly dominant strategies among symmetric allocation rules maximizing a simple utility function.

A Proofs

Because we consider DSIC, we fix an arbitrary θ_{-i} and we omit it from description in the proofs.

A.1 Proof of Lemma 3

Suppose that there exists a type (v_i, δ_i) such that $x_i(v_i, \delta_i) = T$. Hence, $c_i^T(\delta_i) < \infty$ for some $\delta_i \in \Delta$.

Consider an arbitrary $\delta_i' \neq \delta_i$. If $x_i(v_i', \delta_i') = 0$ for all v_i', then its truthful payoff is constantly $\underline{U}_i$. When the bidder of type $\theta_i' = (v_i', \delta_i')$ misreports $\theta_i = (v_i, \delta_i)$ with $v_i > c_i^T(\delta_i)$, then the deviation payoff is $v_i' - p_i^T(\delta_i)$. Hence, the bidder is better off when $v_i' > (1-\delta_i)c_i^T(\delta_i) + \delta_i c_i^B(\delta_i)$, which contradicts DSIC. Hence, there exists v_i' such that $x_i(v_i', \delta_i') \in \{T, B\}$. That is, $c_i^B(\delta_i') < \infty$. It implies $c_i^B(1) = c_i^T(1) < \infty$.

Suppose $\delta_i' < 1$ and that $x_i(v_i', \delta_i') = B$ for all $v_i' > c_i^B(\delta_i')$. When $v_i' > c_i^B(\delta_i')$, the truthful payoff of type (v_i', δ_i') is $\delta_i'(v_i' - c_i^B(\delta_i')) + \underline{U}_i$. When the bidder misreports $\theta_i = (v_i, \delta_i)$ with $v_i > c_i^T(\delta_i)$, then the deviation payoff is $v_i - p_i^T(\delta_i)$. The deviation is profitable if

$$\delta_i'(v_i' - c_i^B(\delta_i')) < v_i' - (1-\delta_i)c_i^T(\delta_i) - \delta_i c_i^B(\delta_i) \Leftrightarrow v_i' > \frac{(1-\delta_i)c_i^T(\delta_i) + \delta_i c_i^B(\delta_i) - \delta_i' c_i^B(\delta_i')}{1-\delta_i'}$$

Therefore, there exists v_i' such that $x_i(v_i', \delta_i') = T$ and $c_i^T(\delta_i') < \infty$. ■

A.2 Proof of Proposition 1

Only If part. Suppose that a mechanism (x, p) is DSIC. Lemmata 1 and 3 show the existence of a cutoff profile $\{c_i^B(\delta), c_i^T(\delta)\}_{\delta \in \Delta}$. Because $\underline{U}_i$ is the constant term added to the payoff of all types, it is without loss of generality to suppose $\underline{U}_i = 0$ below.

Case 1 When $c_i^B(\delta) = c_i^T(\delta)$ $(\forall \delta \in \Delta)$.

Let $p_i^T \equiv c_i^T(\bar{\delta})$ and $p_i^B \equiv \bar{\delta} p_i^T$. The bidder of type $\theta_i = (v, \delta)$ with $v > c_i^T(\delta)$ and $\delta < 1$ pays $c_i^T(\delta)$ for the top position by the envelope condition. Similarly, the bidder of type $\theta_i = (v, 1)$ with $v > c_i^T(1)$ pays $c_i^T(1)$ for the top or bottom. It is clear that DSIC is violated if $c_i^T(\delta) > c_i^T(\delta')$ for some δ and δ', because the bidder of type (v, δ) with $v > c_i^T(\delta')$ is better off lying and reporting (v, δ'). Hence, we have $c_i^T(\delta) = p_i^T$ for all $\delta \in \Delta$ (condition 1). By construction, we have $\bar{\delta} c_i^B(\bar{\delta}) = p_i^B$ (condition 2).

Suppose that $x_i(c_i^B(\delta), \delta) = B$ for some $\delta < p_i^B / p_i^T = \bar{\delta}$. Then, its truthful payoff must be $\delta c_i^B(\delta) - p_i(c_i^B(\delta), \delta) = 0$ because the bidder of the cutoff type is indifferent between obtaining the bottom and nothing. Then, the bidder of type $(c_i^B(\delta), \bar{\delta})$ is better off lying and reporting $(c_i^B(\delta), \delta)$ because the deviation payoff

$$\bar{\delta} c_i^B(\delta) - p_i(c_i^B(\delta), \delta) > 0,$$

whereas its truthful payoff is zero. This is a contradiction; thus, $x_i(c_i^B(\delta), \delta) \neq B$ for all $\delta < \bar{\delta}$ (condition 4).

Condition 3 obviously holds because $p_i^B < p_i^T$ if and only if $\bar{\delta} < 1$.

Case 2 When $c_i^B(\hat{\delta}) < c_i^T(\hat{\delta})$ for some $\hat{\delta} \in \Delta$.

Let $p_i^B \equiv \hat{\delta} c_i^B(\hat{\delta})$ and $p_i^T \equiv (1-\hat{\delta}) c_i^T(\hat{\delta}) + \hat{\delta} c_i^B(\hat{\delta})$. By Lemma 2, the bidder of discount rate $\hat{\delta}$ pays p_i^B when obtaining the bottom and p_i^T when obtaining T. Note that $p_i^B < p_i^T$.

By Lemma 3, $c_i^T(\delta) < \infty$ for all $\delta \in \Delta$. By Lemma 2, a bidder of discount rate $\delta < 1$ pays $p_i^T(\delta) \equiv (1-\delta)c_i^T(\delta) + \delta c_i^B(\delta)$ when obtaining the top T. It is clear that DSIC is violated if $p_i^T(\delta) \neq p_i^T$ because either type $(v, \hat{\delta})$ or (v, δ) is better off lying about their discount rate. Hence, we have

$$(1-\delta)c_i^T(\delta) + \delta c_i^B(\delta) = p_i^T \tag{25}$$

for all $\delta < 1$ (condition 1).

Consider any $\delta \in \Delta \setminus \{0, \hat{\delta}\}$ with $c_i^B(\delta) < c_i^T(\delta)$. The bidder of type $\theta_i = (v, \delta)$ with $v \in (c_i^B(\delta), c_i^T(\delta))$ obtains the bottom B with paying $\delta c_i^B(\delta)$. It is clear that DSIC is violated if $\delta c_i^B(\delta) \neq p_i^B$ by the same reason as the case of the top. Hence, we have

$$\delta c_i^B(\delta) = p_i^B \Leftrightarrow c_i^B(\delta) = \frac{\hat{\delta}}{\delta} c_i^B(\hat{\delta}). \tag{26}$$

(25) and (26) yield

$$(1-\delta)c_i^T(\delta) = (1-\hat{\delta})c_i^T(\hat{\delta}) \Leftrightarrow c_i^T(\delta) = \frac{1-\hat{\delta}}{1-\delta} c_i^T(\hat{\delta}). \tag{27}$$

Given that $c_i^B(\delta)$ and $c_i^T(\delta)$ satisfy (26) and (27), we need

$$\begin{aligned} c_i^B(\delta) < c_i^T(\delta) &\Leftrightarrow \frac{\hat{\delta}}{\delta} c_i^B(\hat{\delta}) < \frac{1-\hat{\delta}}{1-\delta} c_i^T(\hat{\delta}) \\ &\Leftrightarrow \delta > \frac{\hat{\delta} c_i^B(\hat{\delta})}{(1-\hat{\delta})c_i^T(\hat{\delta}) + \hat{\delta} c_i^B(\hat{\delta})} = \frac{p_i^B}{p_i^T}. \end{aligned} \tag{28}$$

Conversely, suppose that $p_i^B/p_i^T < \delta < 1$ and $c_i^B(\delta) = c_i^T(\delta) = p_i^T$. When the bidder of type (p_i^T, δ) misreports $(p_i^T, \hat{\delta})$, the associated payoff is

$$\delta p_i^T - p_i^B > 0 = U(p_i^T, \delta). \tag{29}$$

Hence, the deviation is profitable, and DSIC is violated. Hence, $c_i^B(\delta) < c_i^T(\delta)$ if and only if $p_i^B/p_i^T < \delta < 1$.

Suppose $\bar{\delta} = 1$. The bidder of type $(v, 1)$ with $v > c_i^B(1) = c_i^T(1) = c$ pays c for either the top or bottom. We must have $c = p_i^B$ and $x_i(v, 1) \in \{0, B\}$ because if $x_i(v, 1) = T$ for some $v > c$, the deviation to a type $(v', \hat{\delta})$ with $c_i^B(\hat{\delta}) < v' < c_i^T(\hat{\delta})$ yields a deviation payoff $v - p_i^B > v - p_i^T$. This implies condition 3, and condition 2 is also confirmed.

Now we have $c_i^B(\delta) = c_i^T(\delta) = p_i^T$ for all $\delta < p_i^B/p_i^T$. To have condition 4, suppose that $x_i(c_i^B(\delta), \delta) = B$ for some $\delta < p_i^B/p_i^T$. Then, the associated truthful payoff is

$$\delta c_i^B(\delta) - p_i^B = \delta p_i^T - p_i^B < 0,$$

which violates DSIC because the bidder has a report to earn zero payoff. Hence, $x_i(c_i^B(\delta), \delta) \neq B$ for all $\delta < p_i^B/p_i^T$.

If part. By Lemmata 1–3, we have DSIC in valuation by Myerson (1981). Suppose that a true discount rate is $\delta \geq \hat{\delta} \equiv p_i^B/p_i^T$. It is clear that the deviation of a type (v, δ) to (v', δ'), with $\hat{\delta} \leq \delta' < 1$, is equivalent to the deviation to (v', δ) by LOP. Hence, such a deviation is not profitable. Deviation to $\delta' = 1$ (if possible) is equivalent to a deviation to (v', δ) with $v' \in (c_i^B(\delta), c_i^T(\delta))$, which is not profitable. Deviation to $\delta' < \hat{\delta}$ is not profitable if $v' > c_i^T(\delta')$. Deviation to (v', δ') with $\delta' < \hat{\delta}$ and $v' < c_i^T(\delta')$ induces a zero deviation payoff.

Suppose that a true discount rate is $\delta < \hat{\delta}$. Similarly, deviation to (v', δ') with $\delta' \leq p_i^B/p_i^T$ or $\delta' = 1$ is not profitable. Suppose $p_i^B/p_i^T < \delta' < 1$. The deviation payoff to $v' \in (c_i^B(\delta'), c_i^T(\delta'))$ is

$$\delta v - \delta' c_i^B(\delta') \leq \frac{p_i^B}{p_i^T}(v - p_i^T)$$

Hence, the deviation payoff is negative if $v < p_i^T = c_i^T(\delta)$. When $v > p_i^T$, the truthful payoff is $v - p_i^T$, so that the deviation is not profitable. ■

A.3 Proof of Theorem 1

Let $\phi_h(\cdot) \equiv \phi(\cdot, \delta_h)$ and $\phi_l(\cdot) \equiv \phi(\cdot, \delta_l)$. Suppose that for all bidder $j \neq 1$, $\theta_j = (v_j, \delta_l)$ and $v_2 > v_3 > \cdots > v_I$. In addition, suppose that $v_2 > v_3 \geq r_l \equiv \phi_l^{-1}(0)$.

Consider the efficient allocation rule. If bidder 1 has the discount rate δ_l, the efficient allocation rule is assortative in v_i. Hence, $c_1^B(\delta_l) = v_3$ and $c_1^T(\delta_l) = v_2$. Accordingly, in the efficient allocation rule, we have $c_1^B(\delta_h) = \delta_l v_3/\delta_h$ and $c_1^T(\delta_h) = (1 - \delta_l)v_2/(1 - \delta_h)$ by LOP.

Consider the relaxed problem. Because the allocation rule is efficient in terms of virtual valuation, we have

$$\phi_l(c_1^B(\delta_l)) = \phi_l(v_3), \quad \phi_l(c_1^T(\delta_l)) = \phi_l(v_2),$$

$$\phi_h(c_1^B(\delta_h)) = \frac{\delta_l}{\delta_h}\phi_l(v_3), \quad \phi_h(c_1^T(\delta_h)) = \frac{1-\delta_l}{1-\delta_h}\phi_l(v_2).$$

Hence, by LOP we have

$$\frac{\delta_l}{\delta_h}\phi_l(v) = \phi_h\left(\frac{\delta_l}{\delta_h}v\right),$$

which yields

$$\lambda_l(v) = \frac{\delta_l}{\delta_h}\lambda_h\left(\frac{\delta_l}{\delta_h}v\right). \tag{30}$$

Similarly, we have

$$\lambda_l(v) = \frac{1-\delta_l}{1-\delta_h}\lambda_h\left(\frac{1-\delta_l}{1-\delta_h}v\right). \tag{31}$$

When the relaxed solution is DSIC, (30) and (31) hold and

$$\frac{\delta_l}{\delta_h}\lambda_h\left(\frac{\delta_l}{\delta_h}v\right) = \frac{1-\delta_l}{1-\delta_h}\lambda_h\left(\frac{1-\delta_l}{1-\delta_h}v\right)$$

for all $v \geq r_l$. However, these two equations never hold simultaneously because λ_h is non-decreasing. Therefore, the relaxed solution is not DSIC. ■

A.4 Proof of Theorem 2

Suppose that v_i and δ_i are independently distributed and that the virtual valuation ϕ is affine. Consider the relaxed solution that maximizes the virtual surplus. Because the relaxed solution is efficient in terms of virtual valuations, it satisfies for all θ_{-i},

$$(1-\delta)\phi(c_i^T(\delta)) + \delta\phi(c_i^B(\delta)) = \phi(c_i^T(0)).$$

This induces

$$(1-\delta)c_i^T(\delta) + \delta c_i^B(\delta) = c_i^T(0).$$

Hence, the relaxed solution is DSIC.

Conversely, suppose that the virtual valuation ϕ is not affine. Then, there exist $\alpha \in (0,1)$ and $x, y \in \mathbb{R}_+$ $(x > y \geq \phi^{-1}(0))$, and

$$(1-\alpha)\phi(x) + \alpha\phi(y) \neq \phi\left((1-\alpha)x + \alpha y\right).$$

Then, the LOP condition does not hold when θ_{-i} is such that $v_{-i}^{(1)} = x$, $v_{-i}^{(2)} = y$, $\delta_j = \delta$ for all $j \neq i$. ■

A.5 Proof of Theorem 3

Suppose that $G(v \mid \delta) = G(\delta v \mid 1)$ for all $v \geq r_\delta$. This implies that

$$\lambda_\delta(v) = \frac{g(v \mid \delta)}{1 - G(v \mid \delta)} = \frac{\delta g(\delta v \mid 1)}{1 - G(\delta v \mid 1)} = \delta \lambda_1(\delta v)$$

for all $v \geq r_\delta$. Consider the relaxed solution that maximizes the virtual surplus. Because the relaxed solution is efficient in terms of virtual valuations, it satisfies either

1. $\phi_\delta(c_i^B(\delta)) < \phi_\delta(c_i^T(\delta))$ and $\delta\phi_\delta(c_i^B(\delta)) = \phi_1(c_i^B(1))$, or
2. $\phi_\delta(c_i^B(\delta)) = \phi_\delta(c_i^T(\delta))$ and $\delta\phi_\delta(c_i^T(\delta)) \leq \phi_1(c_i^B(1)) \leq \phi_\delta(c_i^T(\delta))$.

Suppose that the first case holds in the relaxed solution. Then, we have $c_i^B(\delta) < c_i^T(\delta)$. In addition,

$$\begin{aligned} \delta\phi_\delta(c_i^B(\delta)) = \phi_1(c_i^B(1)) \quad &\Leftrightarrow \quad \delta\left(c_i^B(\delta) - \frac{1}{\delta\lambda_1(\delta c_i^B(\delta))}\right) = \phi_1(c_i^B(1)) \\ &\Leftrightarrow \quad \delta c_i^B(\delta) = c_i^B(1). \end{aligned} \tag{32}$$

Suppose that the second case holds in the relaxed solution. Then, we have $c_i^B(\delta) = c_i^T(\delta)$. In addition,

$$\begin{aligned} \delta\phi_\delta(c_i^T(\delta)) &= \delta\left(c_i^T(\delta) - \frac{1}{\delta\lambda_1(\delta c_i^T(\delta))}\right) \\ &= \phi_1(\delta c_i^T(\delta)). \end{aligned} \tag{33}$$

Hence, we have $\delta c_i^T(\delta) \leq c_i^B(1)$. In addition,

$$\phi_\delta(c_i^T(\delta)) = c_i^T(\delta) - \frac{1}{\delta\lambda_1(\delta c_i^T(\delta))} \leq c_i^T(\delta) - \frac{1}{\lambda_1(c_i^T(\delta))} = \phi_1(c_i^T(\delta)). \tag{34}$$

Hence, we have $\phi_1(c_i^B(1)) \leq \phi_\delta(c_i^T(\delta)) \leq \phi_1(c_i^T(\delta))$, so that $c_i^B(1) \leq c_i^T(\delta)$. Therefore, the relaxed solution is DSIC. ■

A.6 Proof of Theorem 4

Suppose $\lambda_1(v) \geq \lambda_0(v)$ for all v. Then, the virtual valuations satisfy for all v,

$$\phi_1(v) \geq \phi_0(v).$$

Consider the relaxed solution that maximizes the virtual surplus. Because the relaxed solution is efficient in terms of virtual valuations, it satisfies

$$\phi_0(c_i^T(0)) \geq \phi_1(c_i^B(1)). \tag{35}$$

Hence, we have

$$c_i^T(0) \geq \phi_0^{-1}\left(\phi_1(c_i^B(1))\right) \geq c_i^B(1),$$

which implies that the relaxed solution is DSIC. ■

A.7 Proof of Theorem 6

Consider an arbitrary type profile other than bidder i, θ_{-i}. When bidder i is absent, let (v_1, δ_1) be the type of the bidder who is assigned the Top, and let (v_2, δ_2) be that of bidder who is assigned the Bottom. Let

$$F_{-i} \equiv F(v_1, v_2, \delta_2).$$

We focus on generic cases in which there is a unique allocation that gives F_{-i} for θ_{-i}. Hence, we suppose $F_{-i} > F(v_2, v_1, \delta_1)$.

The "if" part is obvious because $F(v^T, v^B, \delta^B) = v^T + \delta^B v^B$. What we need to show is the "only if" direction. We will show that for every (v_1, v_2, δ_2), the cutoff functions $c_i^T(\delta_i, \theta_{-i})$ and $c_i^B(\delta_i, \theta_{-i})$ are uniquely determined by DSIC. Then, F must be a monotone transformation of the social surplus because the efficient allocation is implementable in weakly dominant strategy.

In what follows, the cutoff functions are denoted by $c_i^T(\delta_i)$ and $c_i^B(\delta_i)$, omitting θ_{-i}.

Case 1. $\delta_1 \geq \delta_2$. Then, we have $v_1 > v_2$ because $F_{-i} > F(v_2, v_1, \delta_1) \geq F(v_2, v_1, \delta_2)$.

Suppose $\theta_i = (v_i, \delta_2)$ and $v_i < v_2$. Then, $F(v_1, v_i, \delta_2) < F_{-i}$ and $F(v_i, v_2, \delta_2) < F(v_i, v_1, \delta_1) < F(v_2, v_1, \delta_1) < F_{-i}$, so that we have $x_i(\theta) = 0$.[5]

Suppose $v_i > v_2$. Then, $F(v_1, v_i, \delta_2) > F_{-i}$. In addition, the continuity of F implies that there exists $\epsilon > 0$ and

$$F(v_1, v_2 + \epsilon, \delta_2) > F_{-i} > F(v_2 + \epsilon, v_1, \delta_1),$$

[5] Because $F(v_j, v_2, \delta_2) < F(v_j, v_1, \delta_1) < F_{-i}$ and $F(v_1, v_j, \delta_j) < F_{-i}$ for all $j \neq i, 1, 2$, we have $F(v_i, v_j, \delta_j), F(v_j, v_i, \delta_2) < F_{-i}$. Similar arguments apply to all the cases in the proof.

which implies $c_i^T(\delta_2) > c_i^B(\delta_2) = v_2$.

Suppose $\theta_i = (v_i, \delta_1)$ and $v_i < v_1$. Because $F(v_i, v_2, \delta_2) < F(v_i, v_1, \delta_1) < F(v_1, v_i, \delta_1)$, we have $c_i^T(\delta_1) \geq v_1$. If $v_i > v_1$, we have $F(v_i, v_1, \delta_1) > F(v_1, v_i, \delta_1)$ and $c_i^T(\delta_1) = v_1$.

Hence, by LOP, we have $p_i^B = \delta_2 v_2$ and $p_i^T = (1 - \delta_1)v_1 + \delta_2 v_2$. Given the price vector (p_i^B, p_i^T), all the other cutoffs $c_i^T(\delta_i)$ and $c_i^B(\delta_i)$ are uniquely determined.

Case 2. $\delta_1 < \delta_2$ and $v_1 \leq v_2$.

Suppose $\theta_i = (v_i, \delta_1)$. If $v_i < v_1$, we have $F(v_i, v_1, \delta_1) < F(v_i, v_2, \delta_2) < F_{-i}$ and $F(v_2, v_i, \delta_1) < F(v_2, v_1, \delta_1) < F_{-i}$, so that $x_i(\theta) = 0$. If $v_i > v_1$, $F(v_i, v_2, \delta_2) > F_{-i}$ and $c_i^T(\delta_1) \geq v_1$. By continuity of F, for a sufficiently small $\epsilon > 0$, we have $F(v_2, v_1 + \epsilon, \delta_1) < F(v_1, v_2, \delta_2)$, which implies $x_i \neq B$ for $v_i = v_1 + \epsilon$ and $c_i^B(\delta_1) = c_i^T(\delta_1) = v_1$. Hence, we have $p_i^T = v_1$.

Suppose $\theta_i = (v_i, \delta_2)$. If $v_i < v_2$, $F(v_2, v_i, \delta_2) > F(v_i, v_2, \delta_2) > F(v_i, v_j, \delta_j)$ for all $j \neq i, 2$, which implies $x_i(\theta) \neq T$. If $v_i > v_2$, $F(v_i, v_2, \delta_2) > F(v_2, v_i, \delta_2) > F_{-i}$, which implies $x_i(\theta) = T$. Hence, we have $c_i^T(\delta_1) = v_2 > c_i^B(\delta_1)$. By LOP, we have $p_i^T = (1 - \delta_2)v_2 + p_i^B = v_1$, which yields $p_i^B = v_1 - (1 - \delta_2)v_2$. Given the price vector (p_i^B, p_i^T), all the other cutoffs are uniquely determined.

Case 3. $\delta_1 < \delta_2$ and $v_1 > v_2$.

Suppose $\theta_i = (v_i, \delta_1)$. If $v_i < v_1$, $F(v_i, v_2, \delta_2) < F_{-i}$ and $F(v_i, v_1, \delta_1) < F(v_1, v_i, \delta_1)$, which imply $x_i(\theta) \neq T$. Suppose $v_i = v_1 + \epsilon$ and $\epsilon > 0$ is small. Then, $F(v_i, v_2, \delta_2) > F_{-i}$, $F(v_i, v_1, \delta_1) > F(v_1, v_i, \delta_1)$, and $F(v_i, v_2, \delta_2) > F(v_2, v_1 + \epsilon, \delta_1) > F(v_2, v_1, \delta_1)$ by continuity of F. This implies $x_i(\theta) = T$ and $c_i^T(\delta_1) = v_1$. Either $c_i^B(\delta_1) = v_1$ or $c_i^B(\delta_1) < v_1$ holds.

Suppose $\theta_i = (v_i, \delta_2)$. If $v_i < v_2$ $(< v_1)$), we have $F(v_1, v_i, \delta_2) < F_{-i}$, $F(v_i, v_2, \delta_2) < F_{-i}$, and $F(v_i, v_1, \delta_1) < F(v_2, v_1, \delta_1) < F_{-i}$, so that $x_i(\theta) = 0$. Suppose $v_i = v_2 + \epsilon < v_1$ and $\epsilon > 0$ is small. Then, we have $F(v_1, v_i, \delta_2) > F_{-i} > F(v_i, v_2, \delta_2)$ and $F(v_i, v_1, \delta_1) < F_{-i}$ by continuity of F. Hence, we have $c_i^B(\delta_2) = v_2 < c_i^T(\delta_2)$. Therefore, we have $p_i^B = \delta_2 v_2$.

Case 3.1. $\delta_1 v_1 \leq \delta_2 v_2$.

Suppose $c_i^B(\delta_1) < c_i^T(\delta_1)$. Then, if bidder i has a type (v_i, δ_1) and $v_i \in (v_2, v_1)$, their payoff is

$$\delta_1 v_i - p_i^B < \delta_1 v_1 - \delta_2 v_2 \leq 0,$$

which contradicts IC.

Therefore, $c_i^B(\delta_1) = c_i^T(\delta_1) = v_1$ and $p_i^T = v_1$. Given the price vector (p_i^B, p_i^T), all the other cutoffs are uniquely determined.

Case 3.2. $\delta_1 v_1 > \delta_2 v_2$.

Suppose $c_i^B(\delta_1) = c_i^T(\delta_1) = v_1 = p_i^T$. Then, the truthful payoff of type $\theta_i = (v_1 - \epsilon, \delta_1)$ is zero. However, when the bidder deviates and reports $\tilde{\theta}_i = (v_i, \delta_2)$ with $v_i \in (v_2, v_1)$, the associated deviation payoff is

$$\delta_1 v_i - p_i^B = \delta_1 v_1 - \delta_2 v_2 - \delta_1 \epsilon > 0$$

for a sufficiently small $\epsilon > 0$. Hence, it violates IC, so that we must have $c_i^B(\delta_1) < c_i^T(\delta_1)$. By LOP, we have

$$p_i^B = \delta_2 v_2 = \delta_1 c_i^B(\delta_1) \quad \Leftrightarrow \quad c_i^B(\delta_1) = \frac{\delta_2 v_2}{\delta_1} < v_1$$

and

$$p_i^T = (1 - \delta_1) c_i^T(\delta_1) + \delta_1 c_i^B(\delta_1) = (1 - \delta_1) v_1 + \delta_2 v_2.$$

Given the price vector (p_i^B, p_i^T), all the other cutoffs are uniquely determined. ■

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