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Coordinated Strategic Manipulations and Mechanisms in School Choice

Ryo Shirakawa
Graduate School of Economics, the University of Tokyo

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Abstract

In a school choice setting, this paper observes that no group strategy-proof mechanism satisfies even a fairly weak notion of stability. In response to this result, we introduce two monotonicity axioms, which we call *top-dropping monotonicity* and *extension monotonicity*, as alternatives to group strategy-proofness. We prove that these two axioms are equivalent to requirements that no group of students gains from a simple manipulation of their preferences. Then, replacing group strategy-proofness with the two axioms, we find that the Kesten's (2010) *efficiency adjusted deferred acceptance mechanism* is the unique mechanism that satisfies the three criteria. We also provide several applications of the two monotonicity axioms especially for the *deferred acceptance mechanism*.

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1 Introduction

In a school choice problem introduced in Abdulkadiroğlu and Sönmez (2003), a finite set of students needs to be assigned to one school each. Each school has a choice rule, which specifies the students that the school would choose from each set of applicants. Meanwhile, the students have strict preferences over the schools and an outside option. Given a profile of choice rules of the schools, a mechanism determines a matching that

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†The University of Tokyo, Graduate School of Economics. Email: r.shirakawa0723@gmail.com

assigns each student to at most one school, for each reported profile of preferences of the students.

Incentive compatibility of a mechanism, or *strategy-proofness*, is essential to implement intended outcomes. A mechanism is strategy-proof if honest revelation of their preferences is always a weakly dominant strategy for each student. The *deferred acceptance (DA) mechanism* of Gale and Shapley (1962) is one of the leading examples of a strategy-proof mechanism.

If the students are likely to engage in cooperative behavior, we may need a stronger incentive compatibility condition for implementation.[1] A typical example of such a condition is *group strategy-proofness*. A mechanism is group strategy-proof if no group of students can gain by misreporting their preferences. It is known that the DA mechanism is not group strategy-proof in general. Meanwhile, the celebrated *top trading cycles mechanism* of Shapley and Scarf (1974) is group strategy-proof. The other examples include the *serial dictatorship*, shown by Svensson (1999). Pycia and Ünver (2017) provide a comprehensive result, characterizing the full class of efficient and group strategy-proof mechanisms.

Another desirable property that mechanisms should meet is *stability*, which requires that there be no profitable unilateral or pairwise deviation from an output of the mechanism. As Abdulkadiroğlu and Sönmez (2003) point out, stability can be viewed as a requirement of fairness in the context of school choice. Alcalde and Barberà (1994) show that the DA mechanism is the unique strategy-proof mechanism that is stable.

Requiring group strategy-proofness and stability at the same time is too demanding, however. The DA mechanism is not group strategy-proof in general; therefore, existing studies show that no group strategy-proof mechanism is stable. Practitioners face this trade-off: According to Kloosterman and Troyan (2016) and Cerrone et al. (2021) for example, New Orleans Recovery School District once adopted the top trading cycles mechanism, which is efficient and group strategy-proof, but it was eventually abandoned in favor of the DA mechanism due to their fairness concerns.

Building upon these observations, to begin with, this paper clarifies to what extent group strategy-proof mechanisms can have stability properties. If some group strategy-proof mechanisms have "near stability," then adopting such mechanisms will not significantly damage stability. Unfortunately, however, our first result shows a negative conclusion: No group strategy-proof mechanism satisfies even a fairly weak stability requirement, which we call *respecting top-top pairs*.

Respecting top-top pairs requires that a student be matched with his/her most pre-

[1] For instance, this possibility in school choice is reported in Pathak and Sönmez (2008) via an analysis of equilibria in preference revelation games induced from the *Boston mechanism*.

ferred alternative if, the alternative selects him/her among those who prefer the alternative to the outside option. Any stable matching respects top-top pairs, because otherwise the corresponding pair would profitably deviate. This criterion is a slight modification of the concept called *strong top best*, which is first introduced in Chen (2017). We discuss relations between respecting top-top pairs and other stability concepts, in Section 3.

In response to the negative result, that no group strategy-proof mechanism satisfies the rather weak stability requirement, we consider two monotonicity axioms, *top-dropping monotonicity* and *extension monotonicity*, as alternatives to group strategy-proofness. Top-dropping monotonicity requires that an output of a mechanism do not change when, other things being equal, some students who do not match with the most preferred alternative lower the ranking of it. Meanwhile, extension monotonicity requires that an output of a mechanism do not change if, other things being equal, some students who match with a school make some unacceptable schools acceptable.

Each of top-dropping monotonicity and extension monotonicity is weaker than group strategy-proofness. More specifically, we characterize them in terms of robustness to simple coordinated strategic manipulations: A mechanism satisfies top-dropping monotonicity if and only if no group gains by a misreport that differs from their true preferences only in their top choices; and a mechanism satisfies extension monotonicity if and only if no group gains by a misreport that, with some additional conditions, alters the rankings of the outside option from their true preferences. These results resemble the equivalence theorem of Takamiya (2001) between group strategy-proofness and an axiom called *Maskin monotonicity*.

Replacing group strategy-proofness with the two new axioms, our main proposition shows that the *efficiency adjusted deferred acceptance (EADA) mechanism*, introduced in Kesten (2010), is the unique mechanism that satisfies the two monotonicity axioms and respects top-top pairs.[2] Not only does this result resolve the observed conflict, but also it provides a novel axiomatic characterization of the EADA mechanism. It is crucial to understand characteristics of the EADA mechanism also in practice. In support of this claim, Cerrone et al. (2021) report: "in 2019, the Flemish Ministry of Education undertook the first attempt to implement [the EADA mechanism] in the school choice system in Flanders, which is home to more than 68% of the population of Belgium."

Some existing studies characterize the EADA outcomes. Ehlers and Morrill (2020) prove that the EADA outputs the unique efficient element of the unique *legal set*. Tang and Zhang (2021) introduce a notion, *weakly stable*, and characterize the class of EADA

[2] Rigorously speaking, when we call a mechanism the EADA mechanism in this paper, it refers to the EADA mechanism of Kesten (2010) in which all students *consent* to their priorities being violated. See Kesten (2010) for a more detailed explanation.

outcomes incorporating consenting constraints. Reny (2021) shows the EADA algorithm produces the unique *priority-efficient* matching. Our characterization differs from theirs especially in that our axioms focus on the functional structures of mechanisms rather than outcomes, which can be crucial in analyzing students' incentives of manipulations.

On the other hand, some studies characterize the EADA mechanism rather than its outcomes. Dur et al. (2019) prove that the EADA is the unique efficient mechanism that Pareto dominates the DA and provides incentives to consent to their priorities being violated. Doğan and Yenmez (2020) show that the EADA is the unique mechanism that satisfies a "consistency" requirement and Pareto dominates the DA. By comparison, our characterization is by axioms that do not rely on structures of the DA mechanism. Further, our axioms tell us how robust the EADA is to strategic manipulations.

Some papers explore incentive properties of the EADA mechanism, while they do not offer a characterization of it. Troyan and Morrill (2020) and Chen and Möller (2021) respectively show that the EADA mechanism is *not obviously manipulable* and *regret-free truth-telling*, each of which is a weaker criterion than strategy-proofness. Under some special classes of incomplete information, Kesten (2010) and Reny (2021) prove that honest reports constitute an equilibrium. Decerf and Van der Linden (2021) show that the EADA is harder to manipulate than some celebrated mechanisms, such as the *Boston mechanism*. Cerrone et al. (2021) offer the first experimental evidence on the truth-telling rates of the EADA. Our result is unique in that it provides an exact characterization of the EADA mechanism in terms of strategic manipulability.

As applications of the two monotonicity axioms, we demonstrate that the two new monotonicity axioms capture several characteristics of the DA mechanism. First of all, we find that the DA is efficient if and only if it satisfies top-dropping monotonicity. It resembles a result in Kojima and Manea (2010), which shows the equivalence between efficiency and Maskin monotonicity under the DA mechanism. By comparison, our result states that top-dropping monotonicity, a weakening of Maskin monotonicity, is sufficient to imply efficiency. Secondly, we show that the DA is the unique stable mechanism that satisfies extension monotonicity. It is closely related to the results of Kojima and Manea (2010), Morrill (2013) and Chen (2017), each of which pins down the DA among stable mechanisms by some "monotonicity" axioms. These axioms are much stronger than extension monotonicity, and thus our result states that extension monotonicity is sufficient to characterize the DA mechanism. Third, we get a novel axiomatic characterization of the DA mechanism: The DA is the unique mechanism that satisfies strategy-proofness, extension monotonicity, and respects top-top pairs.

Finally, the last two results have an interesting implication beyond the DA mechanism: For any mechanism that satisfies extension monotonicity and respects top-top pairs, it is strategy-proof if and only if it is stable. Kumano (2013) proves this conclusion focusing on the *Boston mechanism*, but our result applies to any other mechanism satisfying extension monotonicity and respecting top-top pairs. In Section 6 we present some mechanisms that meet the two criteria.

The rest of this paper is organized as follows. We introduce the model in Section 2. Section 3 presents an impossibility thoerem. In Section 4, we introduce several monotonicity axioms including new ones. We also discuss relationships between the two new axioms and coordinated strategic manipulability. Section 5 presents our main result. In Section 6, we provide applications of our axioms. Section 7 concludes. The proof for our main result is relegated to Appendix.

2 Model

There are disjoint finite sets of students I and schools S. Each school $s \in S$ has a capacity $q_s \in \mathbb{N}$. There is an outside option $\varnothing$ for the students. A matching $\mu : I \to S \cup \{\varnothing\}$ allocates each student to a school or the outside option with constraints $|\mu^{-1}(s)| \leq q_s$ for each school $s \in S$. We assume that $q_\varnothing = |I|$, that is, the number of available seats of the outside option is not scarce. Let $\mathcal{M}$ be the set of all matchings.

Each student $i \in I$ has a strict preference relation R_i over the set of alternatives $S \cup \{\varnothing\}$. Let P_i be the asymmetric part of R_i, that is, for each $s, s' \in S \cup \{\varnothing\}$, sR_is' if and only if either sP_is' or $s = s'$. Denote by $R = (R_i)_{i \in I}$ a preference profile of the students. We use the notation $R_N = (R_i)_{i \in N}$ for each subset of students $N \subset I$. An alternative $s \in S \cup \{\varnothing\}$ is **acceptable** to a student $i \in I$ at R_i if $sR_i\varnothing$, and **unacceptable** to i at R_i otherwise. For two matchings $\mu, \mu' \in \mathcal{M}$, we write $\mu' R \mu$ when it holds that $\mu'(i) R_i \mu(i)$ for all students $i \in I$. Let $\mathcal{R}$ be the set of all preference profiles.

A choice rule for a school $s \in S$ is a correspondence $C_s : 2^I \to 2^I$ such that $C_s(N) \subset N$ and $|C_s(N)| \leq q_s$ for all subsets of students $N \subset I$. Its interpretation is the following: For each set of applicants $N \subset I$, the school $s \in S$ admits the students in $C_s(N)$ and rejects the students not in $C_s(N)$. Define $C_\varnothing(N) \equiv N$ for each $N \subset I$, which is consistent with the assumption that $q_\varnothing = |I|$. Let $C = (C_s)_{s \in S}$ be a profile of choice rules.

We say that a choice rule C_s is **acceptant** if we have $|C_s(N)| = \min\{q_s, |N|\}$ for all $N \subset I$. A profile of choice rules C is acceptant if C_s is acceptant for all schools $s \in S$. A choice rule C_s is **substitutable** if we have $C_s(N) \cap M \subset C_s(M)$ for all $N \subset I$ and for all $M \subset I$ with $M \subset N$. A profile of choice rules C is substitutable if C_s is substitutable for all schools $s \in S$. One important class of substitutable choice rules is an acceptant **responsive** choice rule C_s for a strict linear order $\succ_s$ over the set of students I, which is defined as follows. For each subset of students $N \subset I$, $C_s(N)$ is the set of $\min\{q_s, |N|\}$ top ranked students in N according to the ordering $\succ_s$. Throughout the paper, we restrict attention to acceptant and substitutable choice rules. Let $\mathcal{C}$ be the set of all profiles of choice rules that are acceptant and substitutable.

Now, we are ready to define mechanisms considered in this paper.

Definition 1. A mechanism $\varphi : \mathcal{R} \times \mathcal{C} \to \mathcal{M}$ is a mapping that assigns a matching for each pair of a preference profile and an acceptant substitutable profile of choice rules.

One of desirable properties of matchings and mechanisms is stability. A matching μ is **individually rational** at a preference profile R if $\mu(i) R_i \varnothing$ holds for all students $i \in I$. A matching μ is **blocked** by a student-school pair (i, s) at a pair of profiles of preferences and choice rules (R, C), if both $s P_i \mu(i)$ and $i \in C_s(\mu^{-1}(s) \cup \{i\})$ hold. Then, we say that a matching μ is **stable** at (R, C) if it is individually rational at R and it is not blocked by any student-school pair at (R, C). A mechanism φ is stable if the matching $\varphi(R, C)$ is stable at (R, C) for all R and C.

Roth and Sotomayor (1992) show that the following **deferred acceptance (DA) algorithm** produces a stable matching for each R and C.

- Step 1. Every student applies to his most preferable alternative under R. Students who apply to the outside option $\varnothing$ match with it. Let N_s^1 be the set of students applying to the school $s \in S$. Each school $s \in S$ tentatively accepts the students in $M_s^1 = C_s(N_s^1)$ and rejects the applicants in $N_s^1 \backslash M_s^1$.

- Step $k (\geq 2)$. Every student who was rejected at the step $k-1$ applies to his next preferable alternative under R. Students who apply to the outside option $\varnothing$ match with it. Let N_s^k be the new set of students applying to the school $s \in S$. Each school tentatively accepts the students in $M_s^k = C_s(M_s^{k-1} \cup N_s^k)$ and rejects the applicants in $(M_s^{k-1} \cup N_s^k) \backslash M_s^k$.

- The algorithm ends at the step when no student is rejected by a school. Each student tentatively accepted by a school at the last step match with the school.

The **deferred acceptance (DA) mechanism** φ^* is a mechanism that maps each (R, C) to the matching obtained when the DA algorithm is applied for (R, C).

Another important property of matchings is efficiency. A matching μ' **Pareto dominates** a matching μ at a preference profile R if we have $\mu' R \mu$ and $\mu' \neq \mu$. We say that a mechanism ψ Pareto dominates a mechanism φ if we have $\psi(R, C) R \varphi(R, C)$ for all R and C, and $\psi \neq \varphi$. If $\psi \neq \varphi$ is not necessarily true, we say that ψ weakly Pareto dominates φ. A matching μ is **efficient** at R if there exists no matching that Pareto dominates μ at R. A mechanism φ is efficient if $\varphi(R, C)$ is efficient at R for all R and C.

Kesten (2010) shows that the DA mechanism can be highly inefficient,[3] and proposes the **efficiency adjusted deferred acceptance (EADA) algorithm**, whose output is efficient and (weakly) Pareto dominates that of the DA algorithm. Bando (2014) and Tang and Yu (2014) design outcome-equivalent mechanisms under acceptant responsible choice rules. Ehlers and Morrill (2020) extend the mechanism of Tang and Yu (2014) to acceptant and substitutable choice rules.

Instead of the Kesten's (2010) original EADA algorithm, we describe the "simplified" algorithm of Ehlers and Morrill (2020), which goes as follows. We say that an alternative $s \in S \cup \{\varnothing\}$ is **underdemanded** at a pair of a preference profile and a matching (R, μ) if $\mu(i) R_i s$ holds for all students $i \in I$.

- Step 1. Run the DA algorithm at (R, C). For each underdemanded alternative $s \in S \cup \{\varnothing\}$ and each student $i \in I$ assigned to s, permanently assign i to s and then remove both i and s.
- Step $k(\geq 2)$. Run the DA algorithm at (R, C) on the remaining population. For each underdemanded alternative $s \in S \cup \{\varnothing\}$ and each student $i \in I$ assigned to s, permanently assign i to s and then remove both i and s.
- The algorithm ends at the step at which all alternatives are removed.

The above algorithm stops within finite steps because, at each step of the algorithm, the last proposers of the DA algorithm always match with underdemanded alternatives. The **efficiency adjusted deferred acceptance (EADA) mechanism** φ^{**} is a mechanism that maps each (R, C) to the matching obtained when the EADA algorithm is applied for (R, C).

3 Impossibility Result

This section describes that no group strategy-proof mechanism satisfies a stability requirement, which we call **respecting top-top pairs**. First of all, we introduce definitions of strategy-proofness and group strategy-proofness. We define the following notion as a preparation, which will be useful especially in Section 4.

Definition 2. For a mechanism φ, a group N **gains by a misreport** R'_N at (R, C) if

- $\varphi(R', C) R_i \varphi(R, C)$ for all $i \in N$; and

[3] To be more precise, Kesten (2010) proves that, for any set of schools and their capacities, there exists a set of students, a preference profile, and a profile of choice rules at which all students match with either the worst or the second-worst alternative under the DA mechanism.

- $\varphi(R', C) P_i \varphi(R, C)$ for some $i \in N$,

where $R' \equiv (R'_N, R_{I \setminus N})$.

Then, a mechanism φ is **strategy-proof** if no singleton gains by a misreport at some (R, C). The DA mechanism φ^* is strategy-proof, but the EADA mechanism φ^{**} is not (See Kesten (2010)). A mechanism φ is **group strategy-proof** if no group of students gains by a misreport at some (R, C). The DA mechanism and the EADA mechanism are not group strategy-proof in general.

Next, let us consider a stability requirement, **respecting top-top pairs**. For each preference profile R and an alternative $s \in S \cup \{\varnothing\}$, let $N_s^R \equiv \{j \in I \mid sR_j\varnothing\}$ be the set of students for whom s is acceptable at R. We may view the set N_s^R as the set of all potential applicants for the alternative s under the profile R. A pair of a student $i \in I$ and an alternative $s \in S \cup \{\varnothing\}$ is a **top-top pair** at (R, C), if sR_is' for all $s' \in S \cup \{\varnothing\}$ and $i \in C_s(N_s^R)$.

Definition 3. A matching μ **respects top-top pairs** at (R, C) if we have $\mu(i) = s$ for any top-top pair (i, s) at (R, C). A mechanism φ **respects top-top pairs** at C if $\varphi(R, C)$ respects top-top pairs at (R, C) for all R. A mechanism φ respects top-top pairs if it respects top-top pairs at all C.

In other words, a matching is said to respect top-top pairs when it always matches top-top pairs. This notion is a slight modification of a requirement called **strong top best**, which is first introduced in Chen (2017). Strong top best requires a matching to match a student i with a school s if (i, s) is a top-top pair. It is weaker than our requirement in that it limits the requirement to student-school pairs. Moreover, strong top best is a strengthening of **mutual best** of Morrill (2013). Therefore, any matching which respects top-top pairs satisfies both strong top best and mutual best.

Yet, our notion is still a rather weak stability requirement. First of all, any stable matching respects top-top pairs, because otherwise, the corresponding pair would form a blocking pair, from substitutability of the choice rules. Secondly, the property of respecting top-top pairs is a strictly weaker requirement than that of several existing weak stability notions. Such examples include **reasonably fairness** of Kesten (2004), **stable-dominating** of Alva and Manjunath (2019), **essential stability** of Troyan et al. (2020), **weak stability** of Tang and Zhang (2021), and **priority-neutrality** of Reny (2021).[4]

Now, we are ready to state the first observation of this paper. The following proposition claims that there exists no group strategy-proof mechanism that respects top-top

[4]We will not explain each concept here. These papers state that the outputs from the EADA mechanism satisfy their stability requirements, instead of being stable.

pairs. Hence, group strategy-proofness is incompatible with all the stability requirements mentioned in the above paragraph.

Proposition 1. *No group strategy-proof mechanism respects top-top pairs.*

Proof. Suppose that there are three students $N = \{i, j, k\}$ and two schools $S = \{s, s'\}$ with capacities $q_s = q_{s'} = 1$. Consider acceptant responsive choice rules C_s and $C_{s'}$ for the following strict linear orders $\succ_s$ and $\succ_{s'}$, respectively.

$\succ_s$	$\succ_{s'}$
k	i
j	j
i	k

Next, consider the following lists of preferences of the three students.

R_i	R'_i	R''_i	R_j	R'_j	R_k	R'_k	R''_k
s	s	s'	s	$\varnothing$	s'	s'	s
s'	$\varnothing$	$\varnothing$	s'	s	s	$\varnothing$	$\varnothing$
$\varnothing$	s'	s	$\varnothing$	s'	$\varnothing$	s	s'

Now, suppose that a mechanism φ satisfies group strategy-proofness and respects top-top pairs. Let $R = (R_i, R_j, R_k)$. If $\varphi(R, C)(i) = \varnothing$, then the student i gains by a misreport R''_i because φ respects top-top pairs, and therefore, we must have $\varphi(R, C)(i) \neq \varnothing$. Likewise, we have $\varphi(R, C)(k) \neq \varnothing$, because otherwise the student k gains by a misreport R''_k. Therefore, there are only two possibilities,

$$\varphi(R, C) = \mu = \begin{pmatrix} i & j & k \\ s' & \varnothing & s \end{pmatrix}, \text{ or } \varphi(R, C) = \mu' = \begin{pmatrix} i & j & k \\ s & \varnothing & s' \end{pmatrix}.$$

If $\varphi(R, C) = \mu$, then the grand coalition I gains by a misreport $R' = (R'_i, R'_j, R'_k)$, because respecting top-top pairs implies that

$$\varphi(R', C) = \mu' = \begin{pmatrix} i & j & k \\ s & \varnothing & s' \end{pmatrix}.$$

Thus, we must have $\varphi(R, C) = \mu'$.

Finally, consider a preference profile $R^* = (R_i, R_j, R'_k)$. Since (j, s) is a top-top pair at (R^*, C), we have $\varphi(R^*, C)(j) = s$. By the assumption, the group $\{j, k\}$ does not gain by the misreport (R_j, R'_k) at R, and thus $\varphi(R^*, C)(k) = \varnothing$ must hold. Nevertheless, it means that the student k gains by a misreport R_k at R^*, which is a contradiction. $\square$

Remark 1. The proof of Proposition 1 remains valid even if we replace respecting top-top pairs with the property of strong top best. It means that no group strategy-proof mechanism satisfies strong top best. On the other hand, our stability requirement, respecting top-top pairs, is crucial to prove our main proposition (Remark 3).

4 Monotonicities of Mechanisms

This section presents several monotonicity axioms. Especially, in response to the conclusion in Proposition 1, we introduce two new monotonicity axioms as alternatives to group strategy-proofness. We also discuss relations between the two new monotonicity axioms and coordinated strategic manipulability of mechanisms.

We say that a preference R'_i is a **monotonic transformation** of a preference R_i at an alternative $s \in S \cup \{\varnothing\}$ if $s' R'_i s$ implies $s' R_i s$ for any $s' \in S \cup \{\varnothing\}$, i.e., any alternative that is ranked above s in R'_i is also ranked above s in R_i. A preference profile R' is a monotonic transformation of a preference profile R at a matching μ, if R'_i is a monotonic transformation of R_i at $\mu(i)$ for each student $i \in I$.

Definition 4. A mechanism φ satisfies **Maskin monotonicity** at C if, for any preference profiles R and R', we have $\varphi(R', C) = \varphi(R, C)$ whenever R' is a monotonic transformation of R at $\varphi(R, C)$. A mechanism φ satisfies Maskin monotonicity if it satisfies Maskin monotonicity at all C.

Takamiya (2001) proves that a mechanism satisfies group strategy-proofness if and only if it satisfies Maskin monotonicity. Therefore, Proposition 1 also states that there exists no mechanism which satisfies Maskin monotonicity and respects top-top pairs.

The DA mechanism does not satisfy Maskin monotonicity because it is not group strategy-proof. Instead of Maskin monotonicity, Kojima and Manea (2010) prove that the DA mechanism satisfies the following **weak Maskin monotonicity**. We frequently use this result of Kojima and Manea (2010) in the proofs of subsequent results.

Definition 5. A mechanism φ satisfies **weak Maskin monotonicity** at C if, for any preference profiles R and R', we have $\varphi(R', C) R' \varphi(R, C)$ whenever R' is a monotonic transformation of R at $\varphi(R, C)$. A mechanism φ satisfies weak Maskin monotonicity if it satisfies weak Maskin monotonicity at all C.

In response to the impossibility result in Section 3, we now introduce two types of new monotonicity axioms, **top-dropping monotonicity** and **extension monotonicity**, as alternatives to group strategy-proofness. First of all, we say that a preference R'_i is a **top-dropping** of a preference R_i if the following condition holds: Let $s^* \in S \cup \{\varnothing\}$ be the most preferred alternative under R_i.

- For any other two alternatives $s, s' \neq s^*$, we have sR_is' if and only if sR'_is'.

In other words, a top-dropping is a transformation of a preference that, other things being equal, alters only the ranking of the most preferred alternative. Note that, if R'_i is a top-dropping of R_i, then R'_i is a monotonic transformation of R_i at all alternatives except for the most preferred one under R_i.

We say that a preference profile R' is a top-dropping of a preference profile R at a matching μ if, for any student i with $R'_i \neq R_i$, R'_i is a top-dropping of R_i and $\mu(i)$ is not the most preferred alternative at R_i.

The following top-dropping monotonicity requires that mechanisms do not change their outputs for any top-dropping.

Definition 6. A mechanism φ satisfies **top-dropping monotonicity** at C if, for any preference profiles R and R', we have $\varphi(R', C) = \varphi(R, C)$ whenever R' is a top-dropping of R at $\varphi(R, C)$. A mechanism φ satisfies top-dropping monotonicity if it satisfies top-dropping monotonicity at all C.

Any Maskin-monotonic mechanism satisfies top-dropping monotonicity, because any top-dropping is a monotonic transformation. Equivalently from Takamiya (2001), any group strategy-proof mechanism satisfies top-dropping monotonicity.

Remark 2. Top-dropping monotonicity is slightly stronger than it looks. To see this, consider a mechanism satisfying top-dropping monotonicity. An iterative use of top-dropping monotonicity shows invariance of the outputs for the following transition of preference profiles: A student lowers the rank of an alternative that she strictly prefers over her original assignment.

Second, we say that a preference R'_i is an **extension** of a preference R_i if the following two conditions are satisfied:

- For any alternative $s \in S \cup \{\varnothing\}$, $sR_i\varnothing$ implies that $sR'_i\varnothing$.
- For any two schools $s', s'' \in S$, we have $s'R_is''$ if and only if $s'R'_is''$.

In other words, an extension is a transformation of a preference that, other things being equal, lowers the ranking of the outside option. Notice that, if R'_i is an extension of R_i, then R'_i is a monotonic transformation of R_i at all schools. When R'_i is an extension of R_i, existing studies may call R_i a **truncation** of R'_i (See, e.g., Roth and Rothblum (1999)).

We say that a preference profile R' is an extension of a preference profile R at a matching μ if, for any student i with $R'_i \neq R_i$, R'_i is an extension of R_i and $\mu(i) \neq \varnothing$.

The following axiom, extension monotonicity, requires that mechanisms do not change their outputs for any extension.

Definition 7. A mechanism φ satisfies **extension monotonicity** at C if, for any preference profiles R and R', we have $\varphi(R', C) = \varphi(R, C)$ whenever R' is an extension of R at $\varphi(R, C)$. A mechanism φ satisfies extension monotonicity if it satisfies extension monotonicity at all C.

Any Maskin-monotonic mechanism satisfies extension monotonicity, because any extension is a monotonic transformation. Moreover, if a mechanism φ is individually rational and satisfies weak Maskin monotonicity, it satisfies extension monotonicity. To see this, take any R and C, and let R' be an extension of R at $\varphi(R, C)$. Since R' is a monotonic transformation of R at $\varphi(R, C)$, weak Maskin monotonicity implies that $\varphi(R', C) R' \varphi(R, C)$, and thus $\varphi(R', C) R \varphi(R, C)$. This also implies that, from individual rationality, R is a monotonic transformation of R' at $\varphi(R', C)$. Therefore, weak Maskin monotonicity shows $\varphi(R, C) R \varphi(R', C)$. Now we have $\varphi(R', C) = \varphi(R, C)$. Hence, φ satisfies extension monotonicity.

4.1 The Monotonicity Axioms and Strategic Manipulabilitiy

In this subsection, we characterize the two new monotonicity axioms, top-dropping monotonicity and extension monotonicity, in terms of coordinated strategic manipulability of mechanisms. These characterizations resemble those of Takamiya (2001) and Bando and Imamura (2016): Takamiya (2001) proves that Maskin monotonicity and group strategy-proofness are equivalent requirements over mechanisms. Similarly, Bando and Imamura (2016) characterize weak Maskin monotonicity in terms of robustness to strategic manipulations by groups of students.

To begin with, the following proposition says that, a mechanism satisfies top-dropping monotonicity if and only if no group of students gains by a misreport that either drops the most preferred alternative or makes an alternative the most preferable.

Proposition 2. *A mechanism φ satisfies top-dropping monotonicity at C if and only if, no group $N \subset I$ gains by a misreport R'_N at (R, C) for some R such that, for each $i \in N$,*

- *R'_i is a top-dropping of R_i; or*
- *R_i is a top-dropping of R'_i.*

Proof. Fix any profile of choice rules $C \in \mathcal{C}$. Then, let $\varphi(R) \equiv \varphi(R, C)$ for each preference profile $R \in \mathcal{R}$, for notational convenience.

To begin with, we prove the "if" direction. Take any student $i \in I$. Let a profile $R' \equiv (R'_i, R_{I\setminus\{i\}})$ be a top-dropping of R at $\varphi(R)$. It is sufficient to show that we have $\varphi(R') = \varphi(R)$, to prove that φ satisfies top-dropping monotonicity. By assumption, we have both $\varphi(R)R_i\varphi(R')$ and $\varphi(R')R'_i\varphi(R)$. Since the latter implies $\varphi(R')R_i\varphi(R)$, we have $\varphi(R')(i) = \varphi(R)(i)$.

If $\varphi(R)(j) \neq \varphi(R')(j)$ holds for some $j \neq i$, then $\{i, j\}$ gains either at R or R' by a misreport (R'_i, R_j) or (R_i, R_j), respectively. They do not happen from the assumption. Hence, it must hold that $\varphi(R) = \varphi(R')$, which ends the proof of the "if" direction.

Now we prove the "only if" direction. Take any preference profile R. Seeking a contradiction, suppose that a group of students $N \subset I$ gains at R by a misreport R'_N that satisfies the condition in the statement. Define $\mu \equiv \varphi(R)$ and $\mu' \equiv \varphi(R'_N, R_{I\setminus N})$, respectively. Take any student $i \in N$ in the group.

First, suppose that R'_i is a top-dropping of R_i. If $\mu(i)$ is not the most preferred alternative for the student $i \in N$ under R_i, then $\varphi(R'_i, R_{I\setminus\{i\}}) = \mu$ from top-dropping monotonicity. Otherwise, we have $\mu'(i) = \mu(i)$ by the assumption that N gains by a misreport R'_N. In this case, we can obtain R_i from R'_i by lowering the rankings of the alternatives that are preferred to $\mu'(i)$ under R'_i. It means that an iterative applications of top-dropping monotonicity yields $\varphi(R'_{N\setminus\{i\}}, R_{I\setminus(N\setminus\{i\})}) = \mu'$ (See Remark 2).

Second, suppose that R_i is a top-dropping of R'_i. If $\mu'(i)$ is not the most preferred alternative for the student $i \in N$ under R'_i, then top-dropping monotonicity implies that $\varphi(R'_{N\setminus\{i\}}, R_{I\setminus(N\setminus\{i\})}) = \mu'$. Otherwise, we have $\mu'(i)R_i\mu(i)$ by the assumption that N gains by a misreport R'_N at R. In this case, we can obtain R'_i from R_i by lowering the rankings of all the alternatives that are preferred to $\mu'(i)$ under R_i. Since $\mu'(i)R_i\mu(i)$, it means that an iterative applications of top-dropping monotonicity yields $\varphi(R'_i, R_{I\setminus\{i\}}) = \mu$.

We have established that either $\varphi(R'_i, R_{I\setminus\{i\}}) = \mu$ or $\varphi(R'_{N\setminus\{i\}}, R_{I\setminus(N\setminus\{i\})}) = \mu'$ holds. By repeating the above arguments for students in N, we finally get $\mu = \mu'$. This is a contradiction, because we assumed that at least one student in the group N has to be strictly better off. $\square$

Secondly, the following proposition implies that a mechanism satisfies extension monotonicity if no group gains by a misreport that alters a ranking of the outside option. However, as the proposition indicates, we need to restrict a class of misreports further to prove the converse statement.[5]

Proposition 3. *A mechanism φ satisfies extension monotonicity at C if and only if, no group $N \subset I$ gains by a misreport R'_N at (R, C) for some R such that, for each $i \in N$,*

- *R'_i is an extension of R_i and $\varphi(R)(i) \neq \varnothing$; or*
- *R_i is an extension of R'_i and $\varphi(R'_N, R_{I\setminus N})(i) \neq \varnothing$; or*

[5]For instance, the DA mechanism satisfies extension monotonicity, but some groups may gain by a misreport that alters a ranking of the outside option. Example 1 illustrates this point: At the preference profile R, The grand coalition gains by reporting R'.

- $R'_i = R_i$.

Proof. Fix $C \in \mathcal{C}$ and let $\varphi(R) \equiv \varphi(R, C)$ for each $R \in \mathcal{R}$ for convenience.

First, we prove the "if" direction. Let $R' \equiv (R'_i, R_{I\setminus\{i\}})$ be an extension of R at $\varphi(R)$. It is sufficient to show that $\varphi(R') = \varphi(R)$ to prove that φ satisfies extension monotonicity. It is trivially true if $R'_i = R_i$. Suppose that $R'_i \neq R_i$. Then, we have $\varphi(R)(i) \neq \varnothing$, by the definition of extensions. Hence, it holds that $\varphi(R) R_i \varphi(R')$ and $\varphi(R') R'_i \varphi(R)$ by the assumption. The former relation implies $\varphi(R) R'_i \varphi(R')$ from $\varphi(R)(i) \neq \varnothing$. Therefore, we have $\varphi(R')(i) = \varphi(R)(i) \neq \varnothing$.

If $\varphi(R)(j) \neq \varphi(R')(j)$ holds for some $j \neq i$, then $\{i, j\}$ gains either at R or R' by a misreport (R'_i, R_j) or (R_i, R_j), respectively. They are not possible by the assumption, because $\varphi(R')(i) = \varphi(R)(i) \neq \varnothing$. Hence, it must hold that $\varphi(R) = \varphi(R')$, which ends the proof of the "if" direction.

Second, we prove the "only if" direction. Consider any group $N \subset I$ and its misreport R'_N that satisfies the condition in the statement. Let $M \subset N$ be the set of students $i \in N$ such that $R'_i \neq R_i$. We show that $\mu' \equiv \varphi(R'_M, R_{I\setminus M}) = \varphi(R) \equiv \mu$.

Take any student $i \in M$. If R'_i is an extension of R_i, we have $\mu(i) \neq \varnothing$ by the condition in the statement. Then, it implies $\varphi(R'_i, R_{I\setminus\{i\}}) = \mu$ from extension monotonicity. If R_i is an extension of R'_i, then $\mu'(i) \neq \varnothing$ holds by the condition. Therefore, extension monotonicity shows $\varphi(R'_{M\setminus\{i\}}, R_{I\setminus(M\setminus\{i\})}) = \mu'$. By repeating this argument for all students in the set M, we get $\mu' = \mu$. This contradicts with the assumption that at least one student in the group N is strictly better off by the manipulation R'_N. $\square$

5 Main Result

This section presents the main result of this paper, which claims that a unique mechanism satisfies top-dropping monotonicity, extension monotonicity, and respects top-top pairs. Furthermore, it coincides with the EADA mechanism of Kesten (2010). We provide the proof in Appendix.

Proposition 4. *A mechanism φ satisfies top-dropping monotonicity, extension monotonicity, and respects top-top pairs if and only if $\varphi = \varphi^{**}$.*

Not only does Proposition 4 resolve the conflict observed in Proposition 1, but also it provides a novel axiomatic characterization for the EADA mechanism under general choice structures. As discussed in Introduction, our axiomatic characterization is different from the other characterizations of the EADA, especially in that our axioms have implications on how robust a mechanism is to some simple strategic manipulations of groups.

Moreover, all of known characterizations, except for that of Ehlers and Morrill (2020), assume acceptant responsive choice rules.

It is worth mentioning that we derive an alternative characterization of the EADA mechanism φ^{**} in Lemma 8 of the appendix. It says that φ^{**} is the unique mechanism that satisfies top-dropping monotonicity and weakly Pareto dominates the DA mechanism. Abdulkadiroğlu et al. (2009), Kesten (2010) and Alva and Manjunath (2019) show that the DA mechanism φ^* is the unique mechanism that weakly Pareto dominates φ^* and is strategy-proof. Correspondingly, Lemma 8 and Proposition 2 find that φ^{**} is the unique mechanism that weakly Pareto dominates φ^* and is robust against group manipulations by top-droppings and "anti"-top-droppings.

Remark 3. The "if" direction of Proposition 4 do not carry over once we weaken respecting top-top pairs to the property of strong top best. For each preference profile R, let $R^{-\varnothing}$ be a profile such that, other things being equal, $sR_i^{-\varnothing}\varnothing$ for all students i and all schools $s \in S$. Then, consider a mechanism φ, where $\varphi(R,C) = \varphi^{**}(R^{-\varnothing},C)$ for each R and C. Now, suppose that $q_s = |I|$ for all schools $s \in S$. Under such capacities, the mechanism φ always matches the students with their most preferred schools, hence it is easy to see that φ satisfies top-dropping monotonicity, extension monotonicity, and strong top best. Yet, it does not respect top-top pairs, because the students never match with the outside option.

The rest of this section verifies that the three axioms in Proposition 4 are independent. That is, the three examples below demonstrate that, for each of these three axioms, there exists a mechanism that does not satisfy the axiom and satisfies the other two.

Example 1. In this example, we see that extension monotonicity and respecting top-top pairs do not imply top-dropping monotonicity in general.

Consider the DA mechanism φ^*, which is a stable mechanism and satisfies weak Maskin monotonicity (Kojima and Manea (2010)). Since φ^* is stable, it respects top-top pairs. Moreover, φ^* satisfies extension monotonicity as discussed in Section 4.

We can see that φ^* fails to satisfy top-dropping monotonicity, however. Suppose that there are three sutdents $I = \{i,j,k\}$ and two schools $S = \{s,s'\}$ with capacities $q_s = q_{s'} = 1$. Consider acceptant responsible choice rules C_s and $C_{s'}$ for the following strict linear orders $\succ_s$ and $\succ_{s'}$, respectively.

$\succ_s$	$\succ_{s'}$
k	i
j	j
i	k

Next, consider the following lists of preferences of the students.

R_i	R_j	R'_j	R_k
s	s	$\varnothing$	s'
s'	$\varnothing$	s	s
$\varnothing$	s'	s'	$\varnothing$

Now, if we run the DA algorithm at the preference profile $R = (R_i, R_j, R_k)$, then it produces the following matching.

$$\varphi^*(R, C) = \begin{pmatrix} i & j & k \\ s' & \varnothing & s \end{pmatrix}.$$

Notice that R'_j is a top-dropping of R_j, and that j does not match with his/her most preferred school s under $\varphi^*(R, C)$. However, it holds that, for $R' = (R_i, R'_j, R_k)$,

$$\varphi^*(R', C) = \varphi^{**}(R, C) = \begin{pmatrix} i & j & k \\ s & \varnothing & s' \end{pmatrix} \neq \varphi^*(R, C),$$

which indicates that φ^* does not satisfy top-dropping monotonicity.

Example 2. The following example shows that top-dropping monotonicity and respecting top-top pairs do not yield extension monotonicity in general.

Consider the **school-proposing DA mechanism**, whose outputs are given by the following algorithm for each input $(R, C) \in \mathcal{R} \times \mathcal{C}$.

- Step 1. Every school $s \in S$ applies to the set of students $C_s(I)$. Students tentatively accept the most preferred school among all acceptable applicants, and reject the rest.
- Step $k(\geq 2)$. Every school $s \in S$ applies to the set of students $C_s(N)$, where N is the set of students who have not rejected s in the earlier steps. Students tentatively accept the most preferred school among all acceptable applicants and the tentatively accepted school, and reject the rest.
- The algorithm ends at the step when no school is rejected by a student. Each school, which is tentatively accepted by a student at the last step, permanently matches with the student. All remaining students match with the outside option.

Let φ^{SDA} be the school-proposing DA mechanism. Since φ^{SDA} is a stable mechanism (See Roth and Sotomayor (1992)), it respects top-top pairs.

Now, we show that the school-proposing DA satisfies top-dropping monotonicity. Let R'_i be a top-dropping of R_i, and suppose that the student i does not match with his/her most preferable alternative s at some (R, C). It means that $s \neq \varnothing$ because φ^{SDA} is individually rational. Moreover, the school s has never applied to the student i during the algorithm. Therefore, under the inputs $R' = (R'_i, R_{I \setminus \{i\}})$ and C, the school-proposing DA algorithm runs in the completely same manner with the algorithm under R and C. Thus, we have $\varphi^{SDA}(R', C) = \varphi^{SDA}(R, C)$. It means that the school-proposing DA mechanism φ^{SDA} satisfies top-dropping monotonicity.

Finally, we see in the following example that φ does not satisfy extension monotonicity. Suppose that $I = \{i, j\}$ and $S = \{s, s'\}$ with capacities $q_s = q_{s'} = 1$. Then, consider acceptant responsible choice rules C_s and $C_{s'}$ for the following strict linear orders $\succ_s$ and $\succ_{s'}$, respectively.

$\succ_s$	$\succ_{s'}$
j	i
i	j

Consider the following lists of preferences of the students.

R_i	R'_i	R_j
s	s	s'
$\varnothing$	s'	s
s'	$\varnothing$	$\varnothing$

If we run the school-proposing DA algorithm to the preference profile $R = (R_i, R_j)$, we obtain the following matching.

$$\varphi^{SDA}(R, C) = \begin{pmatrix} i & j \\ s' & s \end{pmatrix}.$$

Now, it holds that $\varphi^{SDA}(R, C)(i) \neq \varnothing$, and R'_i is an extension of R_i. However, for the preference profile $R' = (R'_i, R_j)$, which is an extension of the profile R at the matching $\varphi^{SDA}(R, C)$, we have

$$\varphi^{SDA}(R, C) = \begin{pmatrix} i & j \\ s & s' \end{pmatrix} \neq \varphi^{SDA}(R, C),$$

hence this example shows that φ^{SDA} does not satisfy extension monotonicity.

Example 3. Consider a "null mechanism" $\varphi^{\varnothing}$ that always assigns the matching μ such that $\mu(i) = \varnothing$ for each student $i \in I$, for all R and for all C. This mechanism satisfies both top-dropping monotonicity and extension monotonicity, because the resulting matchings are the same under any preference profile and any profile of acceptant substitutable choice rules. However, it is trivial to see that this mechanism does not respect top-top pairs. For instance, in a setting $I = \{i\}$ and $S = \{s\}$ with $q_s = 1$, if s is acceptable under a preference R of the student i, then respecting top-top pairs requires that $\varphi^{\varnothing}(R, C)(i) = s$ for all acceptant choice rules C.

6 Applications of the Monotonicity Axioms

Section 4 introduced the two new monotonicity axioms, and Section 5 showed that these monotonicity axioms help characterize the Kesten's (2010) EADA mechanism. In this section, we present further applications of the newly defined axioms focusing on the DA mechanism. We also show that these results have an interesting implication beyond the DA mechanism.

First of all, the next corollary says that top-dropping monotonicity characterizes efficiency of the DA mechanism. Kojima and Manea (2010) show that, under acceptant and substitutable profile of choice rules, the DA mechanism is efficient if and only if it satisfies Maskin monotonicity or group strategy-proofness. Their result is a generalization of Theorem 1 in Ergin (2002), which shows this claim under acceptant responsive choice rules. Compared to these results, the following result states that, instead of Maskin monotonicity, top-dropping monotonicity is sufficient to imply efficiency of the DA mechanism.

Corollary 1. *The DA mechanism φ^* is efficient at $C \in \mathcal{C}$ if and only if φ^* satisfies top-dropping monotonicity at C.*

Proof. First, if φ^* is efficient at $C \in \mathcal{C}$, it satisfies Maskin monotonicity at C from Kojima and Manea (2010). Hence, it satisfies top-dropping monotonicity at C.

Second, suppose that φ^* satisfies top-dropping monotonicity at C. Consider a mechanism φ such that, for each R and C', $\varphi(R, C') = \varphi^*(R, C')$ if $C' = C$ and $\varphi(R, C') = \varphi^{**}(R, C')$ otherwise. Since φ^* is a stable mechanism, it respects top-top pairs. Besides, φ^* satisfies extension monotonicity. Thus, Proposition 4 implies that $\varphi = \varphi^{**}$ holds, which is efficient (Kesten (2010)). Especially, φ is efficient at C, completing the proof. □

Remark 4. We discussed in Section 4 that top-dropping monotonicity is weaker than group strategy-proofness. According to Corollary 1 and the result of Kojima and Manea (2010), however, we have a stronger relationship under the DA mechanism: The equivalence between top-dropping monotonicity and group strategy-proofness.

Secondly, the next proposition states that extension monotonicity characterizes the DA mechanism along with stability. Kojima and Manea (2010) and Morrill (2013) prove that a stable mechanism φ equals the DA mechanism φ^* if and only if it satisfies weak Maskin monotonicity. Chen (2017) states that a weaker axiom called **rank monotonicity** is sufficient to characterize the DA mechanism among all stable mechanisms. Compared to these statements, one can see that extension monotonicity is weaker than both weak Maskin monotonicity and rank monotonicity under individually rational mechanisms. Hence, the next proposition means that extension monotonicity is enough to pin down the DA mechanism among stable mechanisms.

Proposition 5. *A stable mechanism φ satisfies extension monotonicity if and only if it is the DA mechanism φ^*.*

Proof. The "if" direction is obvious. Suppose that a stable mechanism φ satisfies extension monotonicity. Take any R and C.

For each student $i \in I$, construct a preference R'_i from the original preference R_i as follows. If $\varphi^*(R,C)(i) = \varnothing$ holds, define $R'_i \equiv R_i$. If $\varphi^*(R,C)(i) P_i \varnothing$, then R'_i **truncates** all schools that are strictly less preferred to $\varphi^*(R,C)(i)$ from R_i, that is,

- For any school $s \in S$, we have $\varnothing P'_i s$ if and only if $\varphi^*(R,C)(i) P_i s$.
- For any two schools $s, s' \in S$, we have $s R_i s'$ if and only if $s R'_i s'$.

Here, $\varphi^*(R,C)$ is the unique stable matching at (R', C), which is shown in the proof of Lemma 2 of Kojima and Manea (2010). Thus, we have $\varphi(R',C) = \varphi^*(R,C)$, because φ is a stable mechanism.

Moreover, it implies that R is an extension of R' at $\varphi(R',C)$, because $\varphi^*(R',C)(i) \neq \varnothing$ holds for each student $i \in I$ with $R'_i \neq R_i$, and because the relative orderings between all two schools are the same. Therefore, from extension monotonicity of φ, it holds that $\varphi(R,C) = \varphi(R',C) = \varphi^*(R,C)$, which completes the proof. □

Remark 5. Since the DA is the unique stable mechanism that is strategy-proof from Alcalde and Barberà (1994), Proposition 5 shows the equivalence between extension monotonicity and strategy-proofness under stable mechanisms. It is worth noting that they are independent under general mechanisms, although extension monotonicity is implied by group strategy-proofness. We will present examples that (implicitly) demonstrate the independence in the rest of this section.[6]

Third, we find a novel axiomatic characterization for the DA mechanism with our axioms. The following proposition claims that the DA mechanism is the unique mechanism

[6] First, the mechanism in Example 4 shows that strategy-proofness does not necessarily imply extension monotonicity. Second, the *Boston mechanism* serves as an example under which extension monotonicity does not imply strategy-proofness.

that satisfies strategy-proofness, extension monotonicity, and respects top-top pairs. It is closely related to our main result (Proposition 4): If we replace strategy-proofness with top-dropping monotonicity, we get the EADA mechanism instead of the DA mechanism.

Proposition 6. *A mechanism φ satisfies strategy-proofness, extension monotonicity and respects top-top pairs if and only if $\varphi = \varphi^*$.*

Proof. Again, the "if" direction is trivial. and thus, we only prove the "only if" direction. Suppose that a mechanism φ satisfies strategy-proofness, extension monotonicity, and respects top-top pairs. Take any preference profile R and any profile of choice rules C. For each student $i \in I$, define a preference R'_i from R_i as in the proof of Proposition 5. Recall that $\varphi^*(R, C)$ is a stable matching at (R', C).

Here, we show that, for each student $i \in I$ and the alternative $s = \varphi^*(R, C)(i)$, it holds that $i \in C_s(N_s^{R'})$. If $s = \varnothing$, then by the definition of $C_\varnothing$, we have

$$i \in I = N_\varnothing^{R'} = C_\varnothing(N_\varnothing^{R'}).$$

Suppose that $s \neq \varnothing$. Divide the set of the students who weakly prefer the school s to the matching $\varphi^*(R, C)$ into the following two sets.

$$\begin{aligned} M &\equiv \{j \in I \mid s P'_j \varphi^*(R, C)(j)\}, \text{ and} \\ N &\equiv \{j \in I \mid s = \varphi^*(R, C)(j)\}. \end{aligned}$$

It is easy to see that $N_s^{R'} = N \cup M$ by the construction of R'. Since $\varphi^*(R, C)$ is stable at (R', C), for any student $j \in M$, we have $j \notin C_s(N \cup \{j\})$. Moreover, substitutability of C_s shows that $j \notin C_s(N \cup M) = C_s(N_s^{R'})$. Thus, $i \in N = C_s(N_s^{R'})$ holds, because C_s is acceptant.

Now, for each student i, let R''_i be a preference that ranks $s = \varphi^*(R, C)(i)$ at the top. Then, the above discussion shows $i \in C_s(N_s^{R''})$ from $N_s^{R'} = N_s^{R''}$, where $R'' \equiv (R''_i, R'_{I \setminus \{i\}})$. Thus, at R', since φ respects top-top pairs, the student i can match with s by reporting R''_i. Therefore, strategy-proofness implies $\varphi(R', C)(i) R'_i s = \varphi^*(R, C)(i)$. This relation holds for each student, which shows $\varphi(R', C) R' \varphi^*(R, C)$.

Here, since $\varphi(R', C) R' \varphi^*(R, C)$, R is an extension of R' at $\varphi(R')$. To see this, take any student $i \in I$. First, if $\varphi(R', C)(i) = \varnothing$, then $\varnothing R'_i \varphi^*(R, C)(i)$ shows $\varphi^*(R, C)(i) = \varnothing$, because $\varphi^*(R, C)$ is individually rational at (R', C). In which case, we have $R'_i = R_i$ by the construction. Second, for any alternative $s \in S \cup \{\varnothing\}$, $s' R'_i \varnothing$ implies $s' R_i \varphi^*(R, C)(i)$, which implies that $s' R_i \varnothing$. Third, the orderings between all pairs of two schools are the same. Hence, R is an extension of R' at $\varphi(R', C)$. From extension monotonicity, we have $\varphi(R, C) = \varphi(R', C)$.

Finally, note that $\varphi(R,C)R'\varphi^*(R,C)$ implies $\varphi(R,C)R\varphi^*(R,C)$. Since no strategy-proof mechanism Pareto dominates the DA mechanism as shown in Abdulkadiroğlu et al. (2009), we must have $\varphi(R,C)=\varphi^*(R,C)$, which completes the proof. □

The three axioms in Proposition 6 are independent. The EADA mechanism satisfies extension monotonicity and respects top-top pairs as shown in Proposition 4, but it is not strategy-proof. The "null mechanism" in Example 3 is a simple example of mechanisms that satisfy strategy-proofness and extension monotonicity, while it does not respect top-top pairs. The next example provides a mechanism, which shows that strategy-proofness and respecting top-top pairs do not imply extension monotonicity in general.

Example 4. Consider a mechanism φ which outputs the following matching for each R and each C: Take any student $i \in I$. If we have $i \notin C_s(I)$ and $sR_j\varnothing$ for all schools $s \in S$ and for all students $j \neq i$, then $\varphi(R,C)(i) \equiv \varnothing$. If not, $\varphi(R,C)(i) \equiv \varphi^*(R,C)(i)$. Note that $\varphi(R,C)$ is a matching by definition, because the number of students assigned to each school never exceeds that of assigned students under the DA algorithm.

First of all, one can see that the mechanism φ respects top-top pairs, because we have $\varphi(R,C)(i)=\varphi^*(R,C)(i)=s$ if (i,s) is a top-top pair. Second, we show that φ is strategy-proof. If a student i's assignment coincides with the assignment under the DA mechanism, then i does not gain by any misreport, because the DA mechanism is strategy-proof and individually rational. Otherwise, i matches with the outside option no matter what preference i submits. Therefore, φ is strategy-proof.

Third, we verify that this mechanism does not satisfy extension monotonicity. Suppose that $I=\{i,j\}$ and $S=\{s,s'\}$ with capacities $q_s=q_{s'}=1$. Let the schools have acceptant responsible choice rules C_s and $C_{s'}$ for a common strict linear order $i \succ j$. Then, consider the following preferences of the students.

R_i	R'_i	R_j
s	s	s'
$\varnothing$	s'	$\varnothing$
s'	$\varnothing$	s

Under the profile $R=(R_i,R_j)$, the output of the mechanism φ coincides with that of the DA mechanism, which assigns s to i and s' to j. Accordingly, the profile $R'=(R'_i,R_j)$ is an extension of R at $\varphi(R,C)$. However, the mechanism φ matches j with the outside option $\varnothing \neq s'$ at R' by the definition. Thus, the mechanism φ does not satisfy extension monotonicity.

Finally, we find that Proposition 5 and Proposition 6 have an implication for mechanisms beyond the DA mechanism. The next corollary states that, for any fixed profile of choice rules, for any mechanism that satisfies extension monotonicity and respects top-top pairs, its outputs are stable if and only if it is strategy-proof. In other words, the two axioms find a class of mechanisms under which strategy-proofness and stability are equivalent.

Corollary 2. *Fix any $C \in \mathcal{C}$. Suppose that a mechanism φ satisfies extension monotonicity at C and respects top-top pairs at C. Then, the following three are equivalent. (i): The mechanism φ is stable at C; (ii): $\varphi(R, C) = \varphi^*(R, C)$ for all R; and (iii): the mechanism φ is strategy-proof at C.*

Proof. Let φ' be a mechanism such that, for each R and C, $\varphi'(R, C') = \varphi(R, C)$ if $C' = C$ and $\varphi'(R, C') = \varphi^*(R, C')$ otherwise. Note that φ' satisfies extension monotonicity and respects top-top pairs.

If φ is stable at C, then $\varphi' = \varphi^*$ from Proposition 5. Therefore, from the construction, we have $\varphi(\cdot, C) = \varphi'(\cdot, C) = \varphi^*(\cdot, C)$, and thus φ is strategy-proof at C. If φ is strategy-proof at C, we have $\varphi' = \varphi^*$ from Proposition 6. Therefore, from the construction, we have $\varphi(\cdot, C) = \varphi'(\cdot, C) = \varphi^*(\cdot, C)$, and thus φ is stable at C. □

We complete this section by providing some examples of mechanisms under which the equivalence of stability and strategy-proofness holds resorting to Corollary 2.

One example is a family of mechanisms called **application-rejection mechanisms**, which is introduced in Chen and Kesten (2017). Throughout the rest of this section, we fix an acceptant responsive choice rules for a strict linear order profile $\succ$. For each parameter $e \in \mathbb{N}$, which is a strictly positive natural number, the application-rejection mechanism $\varphi^e : \mathcal{R} \to \mathcal{M}$ outputs a matching according to the following **application-rejection algorithm** for each profile R.

- Round $t = 0, 1, \ldots$:
  - Step 1. Each unassigned student from the previous rounds applies to his/her $(te + 1)$th choice at R. Each school tentatively accepts students from the applicants following their order $\succ$ up to their remaining capacities. The rest of the applicants are rejected.
  - Step k, $2 \leq k \leq e$. The rejected students in the previous step apply to their $(te+k)$th choice at R. Each school tentatively accepts students among the pool of the applicants and the tentatively accepted students following their order $\succ$ up to their remaining capacities. The rest of the applicants are rejected.
  - Step $e + 1$. The round t ends and each tentatively accepted student is permanently matched with the alternatives. Then, go to the next round $t + 1$.

- The algorithm terminates when all students are assigned to some alternatives.

Intuitively, the application-rejection algorithm resembles the DA algorithm, but the former algorithm finalizes the assignment at the end of each round. The length of the steps in each round is parameterized by $e \in \mathbb{N}$. This family includes various mechanisms such as the **Boston mechanism** φ^1, the **Shanghai mechanism** φ^2, the **Chinese parallel mechanism** φ^e with $2 \leq e < \infty$, and the DA mechanism $\varphi^\infty = \varphi^*$. Chen and Kesten (2017) state that the DA mechanism is the unique strategy-proof mechanism among application-rejection mechanisms.

For each parameter $e \in \mathbb{N}$, the application-rejection algorithm has the following two properties. (i): When there is a top-top pair, they will permanently match in the first step of the first round; and (ii): The outputs are always individually rational, and all relative rankings below the students' assignments are never considered during the algorithm.

These two observations respectively show that the application-rejection mechanisms respect top-top pairs and satisfy extension monotonicity at acceptant responsive choice rules. Corollary 2 then implies that, for any acceptant responsive choice rule, these mechanisms output stable matchings if and only if they are strategy-proof. Kumano (2013) provides a theorem that incorporates this conclusion by focusing on the Boston mechanism φ^1.

Other examples include a generalization of the application-rejection mechanisms, which incorporate reality. As reported in Abdulkadiroğlu et al. (2005), in real-life applications of these mechanisms, it is often the case that students are only allowed to submit a rank order list of a limited number of schools. Under these constraints, even the DA mechanism can be neither stable nor strategy-proof. This issue is theoretically analyzed by Haeringer and Klijn (2009), Pathak and Sönmez (2013) and Decerf and Van der Linden (2021), for instance.

Considering that scenario, for a positive integer $k > 0$, we define φ^e_k to be the mechanism such that $\varphi^e_k(R) \equiv \varphi^e(R(k))$, where $R(k)$ truncates from R all schools that are ranked strictly lower than the kth alternative. In other words, the mechanism φ^e_k is the application-rejection mechanism φ^e, where students can apply to at most k schools. The same two observations with the above applies to these mechanisms. Therefore, the equivalence between stability and strategy-proofness holds as well.

7 Conclusion

This paper introduced two monotonicity axioms, top-dropping monotonicity and extension monotonicity. We characterized these axioms in terms of robustness to coordinated strategic manipulations. Thus, our monotonicity axioms shed light on a new class of non-strategy-proof mechanisms that are yet robust to some simple coordinated strategic manipulations of preferences. For instance, we saw in Example 2 that the school-proposing DA mechanism satisfies top-dropping monotonicity, while it is not strategy-proof. Our main result provided an axiomatic characterization of the Kesten's (2010) EADA mechanism with these monotonicity axioms.

We explained that the two new monotonicity axioms had several applications for the DA mechanism. Top-dropping monotonicity characterizes efficiency of the DA mechanism, and extension monotonicity pins down the DA mechanism among the set of stable mechanisms. As exemplified in Corollary 2 in Section 6, these axioms can also be useful in examining mechanisms other than the DA and the EADA mechanisms, such as the school-proposing DA mechanism.

Appendix: Proof of Proposition 4

First of all, we introduce a part of Lemma 1 of Tang and Yu (2014), which will be a powerful tool to prove some of the subsequent lemmas. Let R be a preference profile and C be an acceptant substitutable profile of choice rules. Then, we say that a student $i \in I$ is **not Pareto improvable** if, for every matching μ that Pareto dominates the matching from the DA mechanism φ^*, it holds that $\mu(i) = \varphi^*(R, C)(i)$.

Tang and Yu (2014) show that all students matched with underdemanded alternatives are not Pareto improvable. Although they prove this result under acceptant responsive choice rules, its proof can be copied verbatim from them.

Lemma 1 (Tang and Yu (2014))**.** *For any preference profile R and for any acceptant and substitutable profile of choice rules C, all students matched with underdemanded schools at $(R, \varphi^*(R, C))$ are not Pareto improvable.*

We will not explore structures of the simplified EADA algorithm of Ehlers and Morrill (2020) itself. Instead, we will examine a sequence of preference profiles from the following algorithm, which is one of the special classes of algorithms considered in the proof of Theorem 3 of Ehlers and Morrill (2020). Ehlers and Morrill (2020) find that, for each profile R, the following algorithm outputs a profile R^K at which the DA algorithm yields the EADA matching of the original profile R:

- Run the DA algorithm at (R, C). Take any student $i \in I$. If i does not match with an underdemanded alternative at $(R, \varphi^*(R, C))$, let $R_i^1 = R_i$. Otherwise, let R_i^1 be a preference that, other relative orderings being equal, makes all alternatives that are strictly preferred to $\varphi^*(R, C)(i)$ under R_i the worst preferred at R_i^1.

- In general, suppose that R^{k-1} is defined for some $k \geq 2$. Run the DA algorithm at (R, C). Take any student $i \in I$. If i does not match with an underdemanded alternative at $(R^{k-1}, \varphi^*(R^{k-1}, C))$, let $R_i^k = R_i^{k-1}$. Otherwise, let R_i^k be a preference that, other relative orderings being equal, makes all alternatives that are strictly preferred to $\varphi^*(R^{k-1}, C)(i)$ under R_i^{k-1} the worst preferred at R_i^k.
- The procedure ends at the step K at which all alternatives are underdemanded at $(R^K, \varphi^*(R^K, C))$, and $\varphi^{**}(R, C) = \varphi^*(R^K, C)$.

The above algorithm stops within finite steps from Tang and Yu (2014) and Ehlers and Morrill (2020). We say that the sequence of the preference profiles $R^1, \ldots, R^K$ is **induced** by the EADA algorithm.

Before moving on to the proofs, we provide some observations on the above algorithms. Recall that the DA mechanism φ^* satisfies weak Maskin monotonicity (Kojima and Manea (2010)). Therefore, since R^k is a monotonic transformation of R^{k-1} at $\varphi^*(R^{k-1}, C)$,

$$\varphi^*(R^k, C) R^k \varphi^*(R^{k-1}, C), \text{ and thus } \varphi^*(R^k, C) R^{k-1} \varphi^*(R^{k-1}, C),$$

at each step $k = 1, 2, \ldots, K$ with $R^0 = R$.

Moreover, it holds that $\varphi^*(R^k, C) R \varphi^*(R^{k-1}, C)$. To see this, it is enough to show that R^k is also a monotonic transformation of R at $\varphi^*(R^k, C)$, for each step k. Suppose that $R_i^k \neq R_i$ holds for some student i. Then, the student i matches with an underdemanded alternative at some earlier step. This implies that R_i^k ranks $\varphi^*(R^k, C)(i)$ at the top, from Lemma 1 and the construction of the sequence of preferences. Thus, R_i^k is a monotonic transformation of R_i at $\varphi^*(R^k, C)(i)$.

Summarizing, for any profile R and the sequence of profiles $R^1, R^2, \ldots, R^K$ induced by the EADA algorithm, we have the following relations in general.

$$\varphi^{**}(R, C) = \varphi^*(R^K, C) R \varphi^*(R^{K-1}, C) R \cdots R \varphi^*(R^1, C) R \varphi^*(R, C).$$

Now we are ready to proceed to the proof of Proposition 4. The proof is divided into the following sequences of lemmas. First, the next two lemmas verify that the EADA mechanism respects top-top pairs and satisfies extension monotonicity.

Lemma 2. *The EADA mechanism φ^{**} respects top-top pairs.*

Proof. Take any R and C. Suppose (i, s) is a top-top pair at (R, C). The DA mechanism φ^* respects top-top pairs because it is stable, hence $\varphi^*(R, C)(i) = s$. Finally, $\varphi^{**}(R, C) R \varphi^*(R, C)$ implies $\varphi^{**}(R, C)(i) = s$. $\square$

Lemma 3. *φ^{**} satisfies extension monotonicity.*

Proof. Recall that the DA mechanism φ^* satisfies extension monotonicity. Take any preference profile R, and let $\bar{R}$ be an extension of R at $\varphi^{**}(R, C)$. Let $R^1, \ldots, R^K$ and $\bar{R}^1, \ldots, \bar{R}^{\bar{K}}$ be the sequences induced by the EADA algorithm under $R^0 \equiv R$ and $\bar{R}^0 \equiv \bar{R}$, respectively.

First of all, $\bar{R}^0$ is an extension of R^0 at $\varphi^*(R^0, C)$, and thus extension monotonicity implies

$$\mu^0 \equiv \varphi^*(\bar{R}^0, C) = \varphi^*(R^0, C).$$

An alternative $s \in S \cup \{\varnothing\}$ is underdemanded at $(\bar{R}^0, \mu^0)$ if and only if it is underdemanded at (R^0, μ^0). To see this, take any student $i \in I$. If $s = \varnothing$, both $\mu^0(i)\bar{R}^0 s$ and $\mu^0(i)R^0 s$ hold, because φ^* is individually rational. If $s \neq \varnothing$, then $\mu^0(i)\bar{R}_i s$ or $\mu^0(i)R_i s$ implies $\mu^0(i) \neq \varnothing$, because φ^* is individually rational. Hence, we have $\mu^0(i)R_i s$ if and only if $\mu^0(i)\bar{R}^0 s$, because $\bar{R}^0$ is an extension of R^0.

Now, take any student i such that $\bar{R}^1_i \neq R^1_i$. Then, from the construction of $\bar{R}^1$ and R^1, the above paragraph shows both $\bar{R}^1_i = \bar{R}_i$ and $R^1_i = R_i$ hold. Recall that $\bar{R}_i$ is an extension of R_i. Moreover, individual rationality of φ^* shows that $\varnothing$ is underdemanded at (R^0, μ^0), and thus, we have $\mu^0(i) \neq \varnothing$ and

$$\varphi^*(R^1, C)(i) R_i \mu^0(i) P_i \varnothing.$$

It shows that $\varphi^*(R^1, C)(i) \neq \varnothing$. Therefore, $\bar{R}^1$ is an extension of R^1 at $\varphi^*(R^1, C)$. Thus, extension monotonicity implies that

$$\mu^1 \equiv \varphi^*(\bar{R}^1, C) = \varphi^*(R^1, C).$$

Again, we can see that an alternative $s \in S \cup \{\varnothing\}$ is underdemanded at $(\bar{R}^1, \mu^1)$ if and only if it is underdemanded at (R^1, μ^1).

Repeating this procedure, for each $k = 0, 1, \ldots, \min\{K, \bar{K}\}$, we can show that

$$\mu^k \equiv \varphi^*(\bar{R}^k, C) = \varphi^*(R^k, C),$$

and that an alternative $s \in S \cup \{\varnothing\}$ is underdemanded at $(\bar{R}^k, \mu^k)$ if and only if it is underdemanded at (R^k, μ^k). Therefore, we must have $K = \bar{K}$, and thus

$$\varphi^{**}(\bar{R}, C) = \varphi^*(\bar{R}^K, C) = \varphi^*(R^K, C) = \varphi^{**}(R, C),$$

completing the proof. $\square$

The following three lemmas together verify that the EADA mechanism satisfies top-dropping monotonicity. This result, that the EADA mechanism satisfies top-dropping monotonicity, is partly a generalization of Lemma 1 of Reny (2021) and Proposition 1 of Chen and Möller (2021), from acceptant responsive choice rules into acceptant substitutable choice rules. The proofs of their results rely heavily on the assumptions of the acceptant responsive choice rule, and thus cannot be applied as-is in our environment.

Lemma 4. *Suppose that $\bar{R}$ is a monotonic transformation of R at $\varphi^*(R, C)$. Then, any student who matches with an underdemanded alternative at $(R, \varphi^*(R, C))$ also matches with an underdemanded alternative at $(\bar{R}, \varphi^*(\bar{R}, C))$.*

Proof. Take any student $i \in I$, and suppose that $s \equiv \varphi^*(R, C)(i)$ is underdemanded at $(R, \varphi^*(R, C))$. From weak Maskin monotonicity of φ^*, we have

$$\varphi^*(\bar{R}, C)\bar{R}\varphi^*(R, C),$$

which implies $\varphi^*(\bar{R}, C)R\varphi^*(R, C)$. Thus, we have $\varphi^*(\bar{R}, C)(i) = s$ from Lemma 1.

Now, we show that s is underdemanded at $(\bar{R}, \varphi^*(\bar{R}, C))$. Take any student $j \in I$. Then, it holds that $\varphi^*(R, C)(j)R_j s$ by the assumption. Since $\bar{R}$ is a monotonic transformation of R at $\varphi^*(R, C)$, we have $\varphi^*(R, C)(j)\bar{R}_j s$. Thus, we get

$$\varphi^*(\bar{R}, C)(j)\bar{R}_j\varphi^*(R, C)(j)\bar{R}_j s.$$

Therefore, the student i matches with the alternative s at $\varphi^*(\bar{R}, C)$, and s is underdemanded at $(\bar{R}, \varphi^*(\bar{R}, C))$. This is the end of the proof. □

The next lemma shows that the EADA mechanism satisfies the following weaker form of Maskin monotonicity: If a student matches with an underdemanded alternative, the EADA mechanism does not alter the output for any monotonic transformation of her original preference.

Lemma 5. *Suppose that $\bar{R}$ is a monotonic transformation of R at $\varphi^*(R, C)$ such that $\bar{R}_i = R_i$ for all students $i \in I$ who do not match with underdemanded alternatives at $(R, \varphi^*(R, C))$. Then it holds that $\varphi^{**}(\bar{R}, C) = \varphi^{**}(R, C)$.*

Proof. Let $R^1, \ldots, R^K$ and $\bar{R}^1, \ldots, \bar{R}^{\bar{K}}$ be the sequences induced by the EADA algorithm under $R^0 \equiv R$ and $\bar{R}^0 \equiv \bar{R}$, respectively. If $\bar{K} < K$, then define $\bar{R}^k \equiv \bar{R}^{\bar{K}}$ for each $k = \bar{K} + 1, \ldots, K$. For each $k = 0, 1, \ldots, K$, let U^k and $\bar{U}^k$ be the set of students who match with underdemanded alternatives at $(R^k, \varphi^*(R^k, C))$ and at $(\bar{R}^k, \varphi^*(\bar{R}^k, C))$, respectively. One can see that U^k and $\bar{U}^k$ respectively satisfy

$$U^0 \subset U^1 \subset \cdots \subset U^K \text{ and } \bar{U}^0 \subset \bar{U}^1 \subset \cdots \subset \bar{U}^K.$$

We show the following two arguments by mathematical induction, which finally prove that $\bar{R}^K$ is a monotonic transformation of R^K at $\varphi^*(R^K, C)$:

- R^k is a monotonic transformation of $\bar{R}^{k-1}$ at $\varphi^*(\bar{R}^{k-1}, C)$; and
- $\bar{R}^k$ is a monotonic transformation of R^k at $\varphi^*(R^k, C)$,

for each $k = 1, \ldots, K$.

To begin with, we show the above two arguments in the case of $k = 1$. Take any student $i \in I$. First, we prove the former statement. If $i \in U^0$ holds, weak Maskin monotonicity of φ^* and Lemma 1 show that

$$\varphi^*(\bar{R}^0, C)R^0\varphi^*(R^0, C), \text{ and thus } \varphi^*(\bar{R}^0, C)(i) = \varphi^*(R^0, C)(i) \equiv s.$$

Since R^1_i ranks s at the top by the construction, it is a monotonic transformation of $\bar{R}^0_i$ at s. Meanwhile, if $i \notin U^0$ holds, then $R^1_i = R^0_i = \bar{R}^0_i$ by assumption. Hence, these two arguments show that R^1 is a monotonic transformation of $\bar{R}^0$ at $\varphi^*(\bar{R}^0, C)$.

Next, we prove the latter argument with $k = 1$. If $i \notin \bar{U}^0$ holds, then $i \notin U^0$ by Lemma 4, hence we have $\bar{R}^1_i = \bar{R}^0_i = R^0_i = R^1_i$. If $i \in \bar{U}^0$, then from the former argument with $k = 1$, weak Maskin monotonicity of φ^* and Lemma 1 show that

$$\varphi^*(R^1, C)\bar{R}^0\varphi^*(\bar{R}^0, C), \text{ and thus } \varphi^*(R^1, C)(i) = \varphi^*(\bar{R}^0, C)(i) \equiv s.$$

Since $\bar{R}^1_i$ ranks s at the top by the construction, it is a monotonic transformation of R^1_i at s. Therefore, $\bar{R}^1$ is a monotonic transformation of R^1 at $\varphi^*(R^1, C)$.

Now, suppose that the two arguments are true at $k < K$. We show that they are also true at $k + 1$. To see the former argument, take any $i \in I$. If we have $i \notin U^k$, then $i \notin \bar{U}^{k-1}$ from the inductive hypothesis and from the contrapositive of Lemma 4. Hence, we get $R^{k+1}_i = R_i = \bar{R}_i = \bar{R}^k_i$. Suppose that $i \in U^k$. Then, R^{k+1} and $\bar{R}^k$ are monotonic transformations of R^k at $\varphi^*(R^k, C)$ by construction and by the inductive hypothesis, respectively. Therefore, weak Maskin monotonicity of the DA mechanism φ^* and Lemma 1 together imply that

$$\varphi^*(\bar{R}^k, C)(i) = \varphi^*(R^{k+1}, C)(i) = \varphi^*(R^k, C)(i) \equiv s.$$

Since R^{k+1}_i ranks s at the top by the construction, it is a monotonic transformation of $\bar{R}^k_i$

at s. Thus, R^{k+1} is a monotonic transformation of $\bar{R}^k$ at $\varphi^*(\bar{R}^k, C)$.

Next, we verify that the latter argument holds at $k+1$. Take any $i \in I$. If $i \notin \bar{U}^k$, then $i \notin U^k$ from the inductive hypothesis and from the contrapositive of Lemma 4. Therefore, we have $R_i^{k+1} = R_i = \bar{R}_i = \bar{R}_i^{k+1}$. Suppose that $i \in U^k$, then $i \in \bar{U}^k$ from Lemma 4. Hence, Lemma 1 shows that

$$\varphi^*(R^{k+1}, C)(i) = \varphi^*(R^k, C)(i) \equiv s, \text{ and } \varphi^*(\bar{R}^{k+1}, C)(i) = \varphi^*(\bar{R}^k, C)(i) \equiv \bar{s}.$$

Meanwhile, $\bar{R}^k$ is a monotonic transformation of R^k at $\varphi^*(R^k, C)$ by the inductive hypothesis. Thus, weak Maskin monotonicity of φ^* and Lemma 1 imply that $s = \bar{s}$. Since $\bar{R}^{k+1}$ ranks s at the top by the construction, it is a monotonic transformation of $\bar{R}^{k+1}$ at s. Finally, suppose that $i \in \bar{U}^k \setminus U^k$ holds. Then, R^{k+1} and $\bar{R}^{k+1}$ are monotonic transformations of $\bar{R}^k$ at $\varphi^*(\bar{R}^k, C)$ by the former statement at $k+1$ and by the construction, respectively. Therefore, weak Maskin monotonicity of φ^* and Lemma 1 imply

$$\varphi^*(R^{k+1}, C)(i) = \varphi^*(\bar{R}^{k+1}, C)(i) = \varphi^*(\bar{R}^k, C)(i) \equiv s.$$

Since $\bar{R}^{k+1}$ ranks s at the top by the construction, it is a monotonic transformation of R^{k+1} at s. Thus, $\bar{R}^{k+1}$ is a monotonic transformation of R^{k+1} at $\varphi^*(R^{k+1}, C)$.

Here, we have established the two arguments for each $k = 1, \ldots, K$. Especially, the profile $\bar{R}^K$ is a monotonic transformation of R^K at $\varphi^*(R^K, C)$. Moreover, R^K ranks $\varphi^*(R^K, C)$ at the top, because otherwise some schools are not underdemanded. Therefore, weak Maskin monotonicity implies that

$$\varphi^{**}(\bar{R}, C) \bar{R} \varphi^*(\bar{R}^K, C) = \varphi^*(R^K, C) = \varphi^{**}(R, C).$$

Finally, we get $\varphi^{**}(\bar{R}, C) R \varphi^{**}(R, C)$ because $\bar{R}$ is a monotonic transformation of R at $\varphi^{**}(R, C)$. The EADA mechanism is efficient, hence we must have $\varphi^{**}(\bar{R}, C) = \varphi^{**}(R, C)$. This is the end of the proof. □

Lemma 6. *φ^{**} satisfies top-dropping monotonicity.*

Proof. Take any profile R, and suppose that $\bar{R}$ is a top-dropping of R at $\varphi^{**}(R, C)$. Let $R^1, \ldots, R^K$ be the sequences induced by the EADA algorithm under $R^0 \equiv R$. As in the proof of Lemma 5, we define U^k to be the set of students who match with underdemanded alternatives at $(R^k, \varphi^*(R^k, C))$ for each $k = 0, 1, \ldots, K$. Finally, define $\bar{R}^k \equiv (\bar{R}_{I \setminus U^k}, R^k_{U^k})$ for each $k = 0, 1, \ldots, K$.

First, we show that $\bar{R}^k$ is a monotonic transformation of R^k at $\varphi^*(R^k, C)$. Take any student $i \in I$. If $i \in U^k$, then $\bar{R}_i^k = R_i^k$ by the definition of $\bar{R}^k$. Next, suppose

$i \notin U^k$ holds, then we have both $\bar{R}_i^k = \bar{R}_i$ and $R_i^k = R_i$. Since $\bar{R}$ is a top-dropping of R at $\varphi^{**}(R,C)$ and $\varphi^{**}(R,C)R\varphi^*(R^k,C)$ holds, $\bar{R}_i = \bar{R}_i^k$ is a monotonic transformation of $R_i = R_i^k$ at $\varphi^*(R^k,C)(i)$. Therefore, the above two arguments imply that $\bar{R}^k$ is a monotonic transformation of R^k at $\varphi^*(R^k,C)$.

From weak Maskin monotonicity of φ^*, the above paragraph shows that

$$\varphi^{**}(\bar{R}^k,C)\bar{R}^k\varphi^*(\bar{R}^k,C)\bar{R}^k\varphi^*(R^k,C), \text{ and thus } \varphi^{**}(\bar{R}^k,C)R^k\varphi^*(R^k,C),$$

for each $k = 0, 1, \ldots, K$. Especially, from the fact that R^K rank $\varphi^*(R^K,C)$ at their top, the above relation at $k = K$ implies that

$$\varphi^{**}(\bar{R}^K,C) = \varphi^*(R^K,C) = \varphi^{**}(R,C).$$

Therefore, it remains to show that

$$\varphi^{**}(\bar{R},C) = \varphi^{**}(\bar{R}^1,C) = \cdots = \varphi^{**}(\bar{R}^K,C).$$

To show these equivalences, we claim that $\bar{R}^{k+1}$ is a monotonic transformation of $\bar{R}^k$ at $\varphi^*(\bar{R}^k,C)$, for each $k = 0, \ldots, K-1$. Take any $i \in I$. If $i \notin U^k$, then we have $\bar{R}_i^{k+1} = \bar{R}_i = \bar{R}_i^k$ from the definition of $\bar{R}^k$. If $i \in U^k$, then $\bar{R}_i^{k+1} = R_i^{k+1}$. Recall that $\bar{R}^k$ is a monotonic transformation of R^k at $\varphi^*(R^k,C)$ from the second paragraph. Therefore, weak Maskin monotonicity of the DA mechanism, $i \in U^k$, and Lemma 1 show that

$$\varphi^*(\bar{R}^k,C)(i) = \varphi^*(R^k,C)(i) \equiv s.$$

By construction, the preference $\bar{R}_i^{k+1} = R_i^{k+1}$ ranks s at the top. Hence, it is a monotonic transformation of $\bar{R}_i^k$ at s.

From the above paragraph, $\bar{R}^{k+1}$ is a monotonic transformation of $\bar{R}^k$ at $\varphi^*(\bar{R}^k,C)$. The above paragraph also shows that $\bar{R}_i^{k+1} \neq \bar{R}_i^k$ implies $i \in U^k$, for each $i \in I$. Then, Lemma 4 states that, if $\bar{R}_i^{k+1} \neq \bar{R}_i^k$ holds, the student i matches with an underdemanded alternative at $(\bar{R}^k, \varphi^*(\bar{R}^k,C))$. Therefore, we can apply Lemma 5, which shows

$$\varphi^{**}(\bar{R}^{k+1},C) = \varphi^{**}(\bar{R}^k,C).$$

Summarizing, we have established that

$$\varphi^{**}(\bar{R},C) = \varphi^{**}(\bar{R}^1,C) = \cdots = \varphi^{**}(\bar{R}^K,C) = \varphi^*(R^K,C) = \varphi^{**}(R,C),$$

which shows that the EADA mechanism satisfies top-dropping monotonicity. This completes the proof. $\square$

The following lemma is a key to prove the "only if" direction of Proposition 4. If a mechanism satisfies the three axioms, it is a weak Pareto improvement over the DA mechanism. This property is one of the leading characteristics of the EADA mechanism.

Lemma 7. *Suppose that a mechanism φ satisfies top-dropping monotonicity, extension monotonicity, and respects top-top pairs. Then, the mechanism φ weakly Pareto dominates the DA mechanism φ^*.*

Proof. Take any R and any C. For each student $i \in I$, construct a preference R'_i as in the proof of Proposition 5: If $\varphi^*(R,C)(i) = \varnothing$, define $R'_i \equiv R_i$. If $\varphi^*(R,C)(i) P_i \varnothing$, then R'_i truncates all schools which are less preferred to $\varphi^*(R,C)(i)$. Recall that we have $i \in C_s(N_s^{R'})$ for all students $i \in I$ and the alternative $s = \varphi^*(R,C)(i)$, which is shown in the proof of Proposition 6.

Here, we establish that $\varphi(R',C) R' \varphi^*(R,C)$ holds. Suppose on the contrary that there exists a student $i \in I$ such that $\varphi^*(R,C)(i) P'_i \varphi(R',C)(i)$. Define $s \equiv \varphi^*(R,C)(i)$, and let R''_i be a preference that, other relative orderings being equal, makes all schools that are strictly preferred to s under R'_i unacceptable at R''_i. Formally, we have the following two conditions:

- For any alternative $s' \in S \cup \{\varnothing\}$ with $s' P'_i s$, we have $\varnothing R''_i s'$.
- For any two schools $s', s'' \in S$ with $s R'_i s', s''$, we have $s' R'_i s''$ if and only if $s' R''_i s''$.

Note that s is the most preferred alternative under R''_i. Let $R'' = (R''_i, R'_{I \setminus \{i\}})$.

Since φ respects top-top pairs, $\varphi(R'',C)(i) = s$ from $N_s^{R'} = N_s^{R''}$. On the other hand, we assume that $s P'_i \varphi(R',C)(i)$. Hence, an iterative use of top-dropping monotonicity implies that $\varphi(R'',C) = \varphi(R',C)$ holds. This is a contradiction.

Here, $\varphi(R',C) R' \varphi^*(R,C)$ implies that R is an extension of R' at $\varphi(R',C)$ as shown in the proof of Proposition 6. Therefore, extension monotonicity implies that $\varphi(R,C) = \varphi(R',C)$. Hence, it holds that $\varphi(R,C) R' \varphi^*(R,C)$, and thus $\varphi(R,C) R \varphi^*(R,C)$ by the construction of R'. It shows that φ weakly Pareto dominates the DA mechanism. $\square$

The next lemma is itself interesting in that it provides another axiomatic characterization for the EADA mechanism. It states that the EADA mechanism is the unique mechanism that satisfies top-dropping monotonicity and weakly Pareto dominates the DA mechanism. As discussed in Section 5, together with Proposition 2, we have a result that resembles that of Abdulkadiroğlu et al. (2009), Kesten (2010) and Alva and Manjunath (2019). Each of these papers show that the DA mechanism is the unique mechanism that is strategy-proof and weakly Pareto dominates the DA.

Lemma 8. *A mechanism φ satisfies top-dropping monotonicity and weakly Pareto dominates the DA mechanism if and only if $\varphi = \varphi^{**}$ holds.*

Proof. The "if" direction follows from Lemma 6. We prove the "only if" direction. Take any R and any C. Let $R^1, R^2, \dots, R^K$ be the sequence induced by the EADA algorithm.

It is sufficient to show that $\varphi(R, C) = \varphi^*(R^K, C)$ holds. By the assumption, we have $\varphi(R^K, C) R^K \varphi^*(R^K, C)$. Now, all alternatives are underdemended at $(R^K, \varphi^*(R^K, C))$ by the construction. Thus, at the matching $\varphi^*(R^K, C)$, all students match with their most preferred alternatives according to the preference profile R^K. Hence, we must have $\varphi(R^K, C) = \varphi^*(R^K, C)$.

Finally, we show that $\varphi(R, C) = \varphi(R^K, C)$. Let $R^0 \equiv R$ and take any $k = 1, 2, \dots, K$. Then, $\varphi(R^{k-1}, C) R^{k-1} \varphi^*(R^{k-1}, C)$ from the assumption. Hence, Lemma 1 implies that $\varphi(R^{k-1}, C)(i) = \varphi^*(R^{k-1}, C)(i)$ for all students $i \in I$ who match with underdemanded alternatives at $(R^{k-1}, \varphi^*(R^{k-1}, C))$. Therefore, we can iteratively apply top-dropping monotonicity to yield $\varphi(R^k, C) = \varphi(R^{k-1}, C)$.

Summarizing, we have established that

$$\varphi(R, C) = \varphi(R^1, C) = \varphi(R^2, C) = \cdots = \varphi(R^K, C) = \varphi^*(R^K, C) = \varphi^{**}(R, C).$$

This is the end of the proof. □

Proof of Proposition 4. The only if part follows from Lemma 2, Lemma 3 and Lemma 6. Conversely, if a mechanism φ satisfies top-dropping monotonicity, extension monotonicity, and respects top-top pairs, then it weakly Pareto dominates the DA mechanism from Lemma 7. Since φ satisfies top-dropping monotonicity, Lemma 8 then implies that $\varphi = \varphi^{**}$, which completes the proof. □

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