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A Theory of Student Assignment

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INTERDISTRICT SCHOOL CHOICE: A THEORY OF STUDENT ASSIGNMENT†

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Abstract. Interdistrict school choice programs—where a student can be assigned to a school outside of her district—are widespread in the US. We introduce a model of interdistrict school choice and present mechanisms that produce *stable* assignments. We consider four categories of policy goals on assignments and identify when the mechanisms can achieve them. By introducing a novel framework of interdistrict school choice, we provide a new avenue of research in market design.

1. Introduction

School choice is a program that uses preferences of children and their parents over public schools to assign children to schools. It has expanded rapidly in the United States and many other countries in the last few decades.[1] Growing popularity and interest in school choice stimulated research in market design, which has not only studied this problem in the abstract, but also contributed to designing specific assignment mechanisms.

Existing market-design research about school choice is, however, limited to *intradistrict* choice, where each student is assigned to a school only in her own district. In other words, the literature has not studied *interdistrict* choice, where a student can be assigned to a school outside of her district. This is an important limitation for at least two reasons. First,

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[1]For instance, Indicator 1 of an education statistics article for the U.S. (Wang et al., 2019) illustrates that percentage of students attending a "chosen" public school (i.e., a public school located outside the assignment boundary based on the student's residence) has increased from 15 percent to 19 percent from 2007 to 2016. Section 2.4 of Eurydice report of European Commission (Parveva et al., 2020) details school choice in Europe and states that in 11 countries in Europe, under certain conditions, it is possible to opt for a public school other than the one initially assigned.

interdistrict school choice is widespread: some form of it is practiced in 43 U.S. states.[2] Second, as we illustrate in detail below, many policy goals in school choice impose constraints across districts in reality, but the existing literature assumes away such constraints. This omission severely limits our ability to analyze these policies of interest.

In this paper, we propose a model of interdistrict school choice. We study mechanisms and interdistrict admissions rules to assign students to schools under which a variety of policy goals can be established. In our setting, policy goals are defined on the district level—or sometimes even over multiple districts. To facilitate the analysis in this setting, we model the problem as matching with contracts between students and districts in which a contract specifies the particular school within the district that the student attends.[34]

We base our analysis to *stability*, which is widely studied in school choice literature. To define stability in our framework, we assume that each district is endowed with an admissions rule represented by a choice function over sets of contracts. We focus our attention on the student-proposing deferred-acceptance mechanism (SPDA) of Gale and Shapley (1962). In our setting, this mechanism is not only stable but also strategy-proof—i.e., it renders truthtelling a weakly dominant strategy for each student.

In this context, we formalize a number of important policy goals. The first is *individual rationality* in the sense that every student is matched with a weakly more preferred school than the school she is initially matched with (in the absence of interdistrict school choice).[5] This is an important requirement, because if an interdistrict school choice program harms students, then public opposition may occur and the program may not be sustainable. The second policy is what we call *improving student welfare*. With this policy goal, we compare SPDA outcomes in interdistrict and intradistrict school choice and characterize district admissions rules which guarantee that no student is hurt from interdistrict school choice. The third policy is what we call *the balanced-exchange policy*: The number of students that each district receives from the other districts must be the same as the number of students that it sends to the others. Balanced exchange is also highly desired by school districts in practice. This is because each district's funding depends on the number of students that it serves and, therefore, if the balanced-exchange policy is not satisfied, then some districts may lose funding, possibly making the interdistrict school choice program impossible. For each of these policy goals, we identify the necessary and sufficient condition for achieving that goal under SPDA as a restriction on district admissions rules.

[2]See `http://ecs.force.com/mbdata/mbquest4e?rep=OE1705`, last accessed on August 9, 2021.

[3]One might suspect that an interdistrict school choice problem can readily be reduced to an intradistrict problem by relabeling a district as a school. This is not the case because, among other things, which school within a district a student is matched with matters for that student's welfare.

[4]We use the terms *assignment* and *matching* interchangeably for the rest of the paper.

[5]While our model assumes that each student is endowed with an initial school, initial schools play no role except for Section 3.1, so one could start with a model without them.

Last, but not least, we also consider a requirement that there be enough student diversity in each district. In fact, diversity appears to be the main motivation for many interdistrict school choice programs in the United States.[6] To put this into context, we note that the lack of diversity is prevalent under intradistrict school choice programs even though they often seek diversity by controlled-choice constraints.[7] This is perhaps unsurprising given that only residents of the given district can participate in intradistrict school choice and there is often severe residential segregation. In fact, a number of studies such as Rivkin (1994) and Clotfelter (1999, 2011) attribute the majority—as high as 80 percent for some data and measure—of racial and ethnic segregation in public schools to disparities between school districts rather than within school districts. Given this concern, many interdistrict choice programs explicitly list achieving diversity as their main goal.

A case in point is the *Achievement and Integration* (*AI*) *Program* of the Minnesota Department of Education (MDE).[8] Introduced in 2013, the AI program incentivizes school districts for integration. A district is required to participate in this program if the proportion of a racial group in the district is considerably higher than that in a neighboring district. In particular, every year the MDE commissioner analyzes fall enrollment data from every district and, when a district and one of its adjoining districts have a difference of 20 percentage points or higher in the proportion of any group of enrolled *protected students* (American Indian; Asian or Pacific Islander; Hispanic; Black, not of Hispanic origin; and White, not of Hispanic origin), the district with the higher percentage is required to be in the AI program.[9] In the 2019-20 school year, 171 school districts participated in this program. Motivated by Minnesota's AI program, we consider a policy goal requiring that the difference in the proportions of each student type across districts be within a given bound. Then, we provide a necessary and sufficient condition for SPDA to satisfy the diversity policy.

The interdistrict school choice problem provides an instance of our model. Some of our results are applicable to other divided (or fragmented) markets. One concrete example is daycare (nursery school) seat allocation in Japan (Kamada and Kojima, 2020). In Japan, the vast majority of slots in daycare centers are allocated through a centralized matching

[6]We refer to Wells et al. (2009) for a review and discussion of interdistrict integration programs.

[7]Ehlers et al. (2014) cites many examples such as NYC, Chicago, St. Louis, Miami-Dade County, and Jefferson County, where different types of "controlled school choice" are used and diversity is one of the main motivations for such programs.

[8]The official AI program website is `https://education.mn.gov/mde/dse/acint/`, last accessed on August 9, 2021.

[9]In Minnesota's AI program, if the difference in the proportion of protected students at a school is 20 percentage points or higher than another school in the same district, the school with the higher percentage is considered a *racially identifiable school* (RIS) and districts with RIS schools also need to participate in the AI program. In this paper, we focus on diversity issues across districts rather than within districts. Diversity problems within districts are studied in the controlled school choice literature that we discuss below.

mechanism, and it is each municipal government that organizes daycare seat allocation for its residents. There are just over 1,700 municipalities in the country, so each matching market is quite localized and small. Naturally, some parents want to send their children to daycare centers outside their municipality, but at the moment such "inter-district daycare seat allocation" is very rare, and when it is done at all, it is often done at an ad hoc, case-by-case level. Our analysis could shed light on how to improve their matching systems. Other potential applications include the residency matching in the US and Canada where a doctor can participate in the residency match of both countries and various tuition and worker exchange programs (Dur and Ünver, 2019).

A key feature of inter-district choice is that each district lacks the authority to enforce the matching for everyone. For example, a district d cannot force another district d' to admit a resident of district d to a school in district d'. At a conceptual level, how to model a situation in which an inter-district school choice mechanism needs to be organized is a novel challenge. At a technical level, this feature leads to need for considering novel types of inter-district constraints. For example, the constraint that inter-district school choice needs support from each district's residents motivates us to analyze a new type of "individual rationality" constraint (Section 3.2), which requires that every student is made weakly better off under inter-district choice compared to intra-district choice. Similarly, the lack of authority of each district to send its residents elsewhere without a consent of the receiving district motivates us to introduce and analyze the "balanced exchange" constraint analyzed in Section 3.3.

One of our methodological contributions is to present a framework in which districts are endowed with admission rules. One particular example of a district's admission rule is to have each school in the district choose its contracts independently.[10] However, our approach is much more flexible and does not require district admission rules be based on individual schools' choice rules or priorities. For instance, our framework allows us to study more general policies on the district level such as having a gender-balanced district rather than imposing the more strict policy that each school within the district is gender balanced.

Beyond specific results described above, one of our primary contributions is to introduce a framework to study interdistrict school choice. In fact, it is not our intention to claim to have studied all or even most policy goals of interest. On the contrary, our hope is to provide a new framework, thereby facilitating more research in interdistrict school choice and market design.

[10]In such a case, district choice rules can be constructed so that a student is not admitted to more than one school.

Related Literature. Our paper is closely related to the controlled school choice literature that studies student diversity in schools in a given district. Abdulkadiroğlu and Sönmez (2003) introduce a policy that imposes type-specific ceilings on each school. More accommodating policies using soft floor constraints—reserves—rather than type-specific ceilings have been proposed and analyzed by Hafalir et al. (2013) and Ehlers et al. (2014).[11] Echenique and Yenmez (2015) provide an axiomatic characterization of such affirmative action policies and provide comparative statics for student welfare.[12] In addition to sharing the motivation of achieving diversity, our paper is related to this literature in that we extend the type-specific floor and ceiling constraints to district admissions rules. In contrast to this literature, however, our policy goals are imposed on districts rather than individual schools, which makes our model and analysis different from the existing ones.

The feature of our paper that imposes constraints on sets of schools (i.e., districts), rather than individual schools, is shared by several recent studies in matching with constraints. Kamada and Kojima (2015) study a model where the number of doctors who can be matched with hospitals in each region has an upper bound constraint. Variations and generalizations of this problem are studied by Goto et al. (2014, 2017), Biro et al. (2010), and Kamada and Kojima (2017, 2018), among others. While sharing the broad interest in constraints, these papers are different from ours in three major respects. First, they do not assume a set of hospitals is endowed with a well-defined choice function, while each school district has a choice function in our model. Second, the policy issues studied in these papers and those studied in ours are different given differences in the intended applications. In particular, constraints in those papers do not address policies involving heterogenous types of doctors and neither are doctors initially endowed with hospitals or regions. In contrast, heterogenous student types and initial matching play a crucial role in our analysis. Third, the techniques used in these papers and ours are quite different. Our approach is mainly an axiomatic (characterization) approach. These differences render our analysis distinct from those of the other papers, with none of their results implying ours and vice versa.

In our model, individual rationality requires that each student be assigned to a school that she weakly prefers to her initial school, as in the literature that studies reallocation of objects to individuals with an initial endowment. This is an active area of study with a variety of applications, e.g., Shapley and Scarf (1974), Abdulkadiroğlu and Sönmez (1999), Guillen and Kesten (2012), Pereyra (2013), and Combe et al. (2016). In contrast to this

[11]Ehlers (2007) is the first paper to study floor constraints, which is incorporated in Ehlers et al. (2014).

[12]In addition to the works discussed above, recent studies on controlled school choice and other two-sided matching problems with diversity concerns include Abdulkadiroğlu (2005), Ergin and Sönmez (2006), Kojima (2012), Westkamp (2013), Dur et al. (2014), Fragiadakis et al. (2015), Kominers and Sönmez (2016), Fragiadakis and Troyan (2017), Nguyen and Vohra (2017), Yenmez (2018), and Dur et al. (2020).

literature, we provide a characterization of district admissions rules so that SPDA satisfies individual rationality, which is not studied in this earlier literature.

One of the notable features of our model is that district admissions rules do not necessarily satisfy the standard assumptions in the literature, such as *substitutability*, which guarantee the existence of a stable matching. In fact, even a seemingly reasonable district admissions rule may violate substitutability because a district can choose at most one contract associated with the same student—namely just one contract representing one school that the student can attend. Rather, we make weaker assumptions following the approach of Hatfield and Kominers (2014).

There is also a recent literature on segmented matching markets in a given district. Manjunath and Turhan (2016) study a setting where different clearinghouses can be coordinated, but not integrated in a centralized clearinghouse, and show how a stable matching can be achieved. In a similar setting, Dur and Kesten (2018) study sequential mechanisms and show that these mechanisms lack desired properties. In another work, Ekmekci and Yenmez (2019) study the incentives of a school to join a centralized clearinghouse. In contrast to these papers, we study which interdistrict school choice policies can be achieved when districts are integrated.

At a high level, the present paper is part of research in market design under various constraints. Real-life auction problems often feature constraints (Milgrom, 2009), and a great deal of attention was paid to cope with complex constraints in a recent FCC auction for spectrum allocation (Milgrom and Segal, 2020). Auction and exchange markets under constraints are analyzed by Bing et al. (2004), Gul et al. (2018), and Kojima et al. (2020). Handling constraints is also a subject of a series of papers on probabilistic assignment mechanisms (Budish et al., 2013; Che et al., 2013; Pycia and Ünver, 2015; Akbarpour and Nikzad, 2020; Nguyen et al., 2016). Closer to ours are Dur and Ünver (2019) and Dur et al. (2015). They consider the balance of incoming and outgoing members—a requirement that we also analyze—while modeling exchanges of members of different institutions under constraints. Although the differences in the model primitives and exact constraints make it impossible to directly compare their studies with ours, these papers and ours clearly share broad interests in designing mechanisms under constraints.

The rest of the paper is organized as follows. Section 2 introduces the model, properties of admission rules/mechanisms, policy goals, and the student-proposing deferred-acceptance algorithm. In Section 3, we study when the policy goals can be satisfied together with stability. More specifically, Sections 3.1 and 3.2 study two different versions of individual rationality, Section 3.3 studies balanced exchange, and Section 3.4 studies diversity. Section 4 concludes. An additional result, omitted proofs, and an example are presented in the Appendices.

2. Model

In this section, we introduce our concepts and notation.

2.1. Preliminary Definitions. There exist finite sets of students $\mathcal{S}$, districts $\mathcal{D}$, and schools $\mathcal{C}$. Each student s and school c has a home district denoted by $d(s)$ and $d(c)$, respectively. Each student s has a type $\tau(s)$ that can represent different aspects of the student such as the gender, race, socioeconomic status, etc. The set of all types is finite and denoted by $\mathcal{T}$. Each school c has a capacity q_c, which is the maximum number of students that the school can enroll. There exist at least two school districts with one or more schools. For each district d, k_d is the number of students whose home district is d. In each district, schools have sufficiently large capacities to accommodate all students from the district, i.e., for every district d, $k_d \leq \sum_{c:d(c)=d} q_c$. For each type t, k^t is the number of type-t students.

We model interdistrict school choice as a matching problem between students and districts. However, merely identifying the district with which a student is matched leaves the specific school she is enrolled in unspecified. To specify which school within a district the student is matched with, we use the notion of contracts: A contract $x = (s, d, c)$ specifies a student s, a district d, and a school c within this district, i.e., $d(c) = d$.[13] For any contract x, let $s(x)$, $d(x)$, and $c(x)$ denote the student, district, and school associated with this contract, respectively. Let $\mathcal{X} \equiv \{(s, d, c)|d(c) = d\}$ denote the set of all contracts. For any set of contracts X, let X_s denote the set of all contracts in X associated with student s, i.e., $X_s = \{x \in X|s(x) = s\}$. Similarly, let X_d and X_c denote the sets of all contracts in X associated with district d and school c, respectively.

Each district d has an ***admissions rule*** that is represented by a choice function Ch_d. Given a set of contracts X, the district chooses a subset of contracts associated with itself, i.e., $Ch_d(X) = Ch_d(X_d) \subseteq X_d$.

Each student s has a strict preference order P_s over all schools and the outside option of being unmatched, which is denoted by $\emptyset$. Likewise, P_s is also used to rank contracts associated with s. Furthermore, we assume that the outside option is the least preferred outcome, so for every contract x associated with s, $x \; P_s \; \emptyset$.[14] The corresponding weak order is denoted by R_s. More precisely, for any two contracts x, y associated with s, $x \; R_s \; y$ if $x \; P_s \; y$ or $x = y$.

A ***matching*** is a set of contracts. A matching X is ***feasible for students*** if there exists at most one contract associated with every student in X. A matching X is ***feasible*** if it is feasible for students and the number of contracts associated with every school in X is

[13]For ease of exposition, a contract will sometimes be denoted by a pair (s, c) with the understanding that the district associated with the contract is the home district of school c.

[14]This assumption plays a role only in Theorems 3 and 4, which is common in the literature on controlled school choice, e.g., Ehlers et al. (2014) and Fragiadakis and Troyan (2017).

at most its capacity, i.e., for any $c \in \mathcal{C}$, $|X_c| \leq q_c$. We assume that there exists a feasible ***initial matching*** $\tilde{X}$ such that every student has exactly one contract.[15] For any student s, if $\tilde{X}_s = \{(s,d,c)\}$ for some district d and school c, then c is called the ***initial school*** of s. The initial matching allows us to formally define one of the policy goals, *individual rationality*, that we study below. It plays no role when individual rationality is not imposed in the analysis.

A ***problem*** is a tuple $(\mathcal{S}, \mathcal{D}, \mathcal{C}, \mathcal{T}, \{d(s), \tau(s), P_s\}_{s \in \mathcal{S}}, \{Ch_d\}_{d \in \mathcal{D}}, \{d(c), q_c\}_{c \in \mathcal{C}}, \tilde{X})$. In what follows, we assume that all the components of a problem are publicly known except for student preferences. Therefore, we sometimes refer to a problem by the student preference profile which we denote as $P_{\mathcal{S}}$. The preference profile of a subset of students $S \subseteq \mathcal{S}$ is denoted by P_S.

2.2. Properties of Admissions Rules. A district admissions rule Ch_d is ***feasible*** if it always chooses a feasible matching. It is ***acceptant*** if, for any contract x associated with district d and matching X that is feasible for students, if x is rejected from X, then at $Ch_d(X)$, either

- the number of students assigned to school $c(x)$ is equal to $q_{c(x)}$, or
- the number of students assigned to district d is at least k_d.

In words, when a district admissions rule is acceptant, a contract $x = (s,d,c)$ can be rejected by district d from a set which is feasible for students only if either the capacity of school c is filled or district d has accepted at least k_d students. Equivalently, if neither of these two conditions is satisfied, then the district has to accept the student.

A district admissions rule satisfies ***substitutability*** if, whenever a contract is chosen from a set, it is also chosen from any subset containing that contract (Kelso and Crawford, 1982; Roth, 1984). More formally, a district admissions rule Ch_d satisfies substitutability if, for every $x \in X \subseteq Y \subseteq \mathcal{X}$ with $x \in Ch_d(Y)$, it must be that $x \in Ch_d(X)$. A district admissions rule satisfies ***the law of aggregate demand*** (LAD) if the number of contracts chosen from a set is weakly greater than that of any of its subsets (Hatfield and Milgrom, 2005). Mathematically, a district admissions rule Ch_d satisfies LAD if, for every $X \subseteq Y \subseteq \mathcal{X}$, $|Ch_d(X)| \leq |Ch_d(Y)|$.[16] A ***completion*** of a district admissions rule Ch_d is another admissions rule Ch'_d such that for every matching X either $Ch'_d(X)$ is equal to $Ch_d(X)$ or $Ch'_d(X)$ is not feasible for students (Hatfield and Kominers, 2014).

[15] In Section 3.2, we also consider the case when the initial matching for each district is constructed using student preferences and district admissions rules.

[16] Alkan (2002) and Alkan and Gale (2003) introduce related monotonicity conditions.

Throughout the paper, we assume that district admissions rules are feasible and acceptant and have completions that satisfy substitutability and LAD.[17,18] Substitutability and LAD are substantial assumptions and would restrict the types of diversity policy goals considered in this paper. However, we think that these assumptions are very important for us to make a progress, because they are sufficient (and quite close to being necessary) for the existence of a stable matching and strategy-proofness of SPDA (detailed in Section 2.4.) To the extent that we accept–consistently with the vast majority of research–that stability and strategy-proofness are important properties, we think that it is interesting to focus on environments in which those assumptions are satisfied.

Next, we provide an example of district admissions rules that satisfy these important properties. This example is referred back in the remainder of the paper while giving practical examples of the choice rules for the four policy goals that we analyze.

2.2.1. *An Example of Admissions Rule.* Consider a district d with schools $c_1, \ldots, c_n$. Each school c_i has an admissions rule Ch_{c_i} such that, for any set of contracts X, $Ch_{c_i}(X) = Ch_{c_i}(X_{c_i}) \subseteq X_{c_i}$. District d's admissions rule Ch_d is defined as follows. For any set of contracts X,

$$Ch_d(X) = Ch_{c_1}(X) \cup Ch_{c_2}(X \setminus Y_1) \cup \ldots \cup Ch_{c_n}(X \setminus Y_{n-1}),$$

where Y_i for $i = 1, \ldots, n-1$ is the set of all contracts in X associated with students who have contracts in $Ch_{c_1}(X) \cup \ldots \cup Ch_{c_i}(X \setminus Y_{i-1})$ and $Y_0 = \emptyset$. In words, we order the schools and let schools choose in that order. Furthermore, if a student is chosen by some school, we remove all contracts associated with this student for the remaining schools.

We now analyze when district admissions rule Ch_d satisfies our assumptions. Specifically, we establish the following results, where the proofs are relegated to Appendix C.

Claim 1. *Suppose that for every school c_i and matching X, $|Ch_{c_i}(X)| \leq q_{c_i}$. Then district admissions rule Ch_d is feasible.*

Claim 2. *Suppose that for every school c_i and matching X, $|Ch_{c_i}(X)| = \min\{q_{c_i}, |X_{c_i}|\}$. Then district admissions rule Ch_d is acceptant.*

Next we study when district admissions rule Ch_d has a completion that satisfies substitutability and LAD. Consider the following district admissions rule Ch'_d: For any set of contracts X,

$$Ch'_d(X) = Ch_{c_1}(X) \cup \ldots \cup Ch_{c_n}(X).$$

Claim 3. *Suppose that for every school c_i, Ch_{c_i} satisfies substitutability and LAD. Then district admissions rule Ch'_d is a completion of Ch_d, and it satisfies substitutability and LAD.*

[17]In Section 3.4, we assume a weaker notion of acceptance when the admissions rule limits the number of students of each type that the district can accept.

[18]Hatfield and Kojima (2010) introduce other notions of weak substitutability.

All of the assumptions on school admissions rules stated in Claims 1, 2, and 3 are satisfied when school admissions rules are ***responsive***: each school has a ranking of contracts associated with itself and, from any given set of contracts, each school chooses contracts with the highest rank until the capacity of the school is full or there are no more contracts left.[19] Responsive admissions rules satisfy substitutability and LAD. Furthermore, for every school c_i, $|Ch_{c_i}(X)| = \min\{q_{c_i}, |X_{c_i}|\}$. By the claims stated above, when school admissions rules are responsive, district admissions rule Ch_d is feasible and acceptant, and it has a completion that satisfies substitutability and LAD.

2.3. Matching Properties, Policy Goals, and Mechanisms. A feasible matching X satisfies ***individual rationality*** if every student weakly prefers her outcome in X to her initial school, i.e., for every student s, $X_s \; R_s \; \tilde{X}_s$.

A ***distribution*** $\xi \in \mathbb{Z}_+^{|\mathcal{C}|\times|\mathcal{T}|}$ is a vector such that the entry for school c and type t is denoted by ξ_c^t. The entry ξ_c^t is interpreted as the number of type-t students in school c at ξ. Furthermore, let $\xi_d^t \equiv \sum_{c:d(c)=d} \xi_c^t$, which is interpreted as the number of type-t students in district d at ξ. Likewise, for any feasible matching X, the ***distribution associated with*** X is $\xi(X)$ whose c,t entry $\xi_c^t(X)$ is the number of type-t students assigned to school c at X. Similarly, $\xi_d^t(X)$ denotes the number of type-t students assigned to district d at X.

We represent a distributional policy goal Ξ as a set of distributions. The policy that each student is matched without assigning any school more students than its capacity is denoted by Ξ^0, i.e., $\Xi^0 \equiv \{\xi | \sum_{c,t} \xi_c^t = \sum_d k_d \text{ and } q_c \geq \sum_t \xi_c^t \text{ for all } c\}$. A matching X ***satisfies the policy goal*** Ξ if the distribution associated with X is in Ξ.

A matching X is ***stable*** (Gale and Shapley, 1962) if it is feasible and

- districts would choose all contracts assigned to them, i.e., $Ch_d(X) = X_d$ for every district d, and
- there exist no student s and no district d who would like to match with each other, i.e., there exists no contract $x = (s, d, c) \notin X$ such that $x \; P_s \; X_s$ and $x \in Ch_d(X \cup \{x\})$.

A ***mechanism*** ϕ takes a profile of student preferences as input and produces a feasible matching. The outcome for student s at the reported preference profile $P_{\mathcal{S}}$ under mechanism ϕ is denoted as $\phi_s(P_{\mathcal{S}})$. A mechanism ϕ satisfies ***strategy-proofness*** if no student can misreport her preferences and get a strictly more preferred contract. More formally, for every student s and preference profile $P_{\mathcal{S}}$, there exists no preference P'_s such that $\phi_s(P'_s, P_{\mathcal{S}\setminus\{s\}}) \; P_s \; \phi_s(P_{\mathcal{S}})$. For any property on matchings, a mechanism satisfies the property if, for every preference profile, the matching produced by the mechanism satisfies the property.

[19]See Chambers and Yenmez (2018) for a characterization of responsive admissions rules using substitutability.

2.4. Student-Proposing Deferred Acceptance Algorithm. To achieve stable matchings with desirable properties, we use a generalization of the deferred-acceptance algorithm of Gale and Shapley (1962).

Student-Proposing Deferred Acceptance Algorithm.

Step 1: Each student s proposes a contract (s, d, c) to district d where c is her most preferred school. Let X_d^1 denote the set of contracts proposed to district d. District d tentatively accepts contracts in $Ch_d(X_d^1)$ and permanently rejects the rest. If there are no rejections, then stop and return $\cup_{d \in \mathcal{D}} Ch_d(X_d^1)$ as the outcome.

Step n ($\mathbf{n > 1}$): Each student s whose contract was rejected in Step $n-1$ proposes a contract (s, d, c) to district d where c is her next preferred school. If there is no such school, then the student does not make any proposals. Let X_d^n denote the union of the set of contracts that were tentatively accepted by district d in Step $n-1$ and the set of contracts that were proposed to district d in Step n. District d tentatively accepts contracts in $Ch_d(X_d^n)$ and permanently rejects the rest. If there are no rejections, then stop and return $\cup_{d \in \mathcal{D}} Ch_d(X_d^n)$ as the outcome.

The student-proposing deferred acceptance mechanism (SPDA) takes a profile of student preferences as input and produces the outcome of this algorithm at the reported student preference profile. When district admissions rules have completions that satisfy substitutability and LAD, SPDA is stable and strategy-proof (Hatfield and Kominers, 2014).

We illustrate SPDA using the following example. We come back to this example later to study the effects of different admissions polices on interdistrict school choice.

Example 1. Consider a problem with two school districts, d_1 and d_2. District d_1 has school c_1 with capacity one and school c_2 with capacity two. District d_2 has school c_3 with capacity two. There are four students: students s_1 and s_2 are from district d_1, whereas students s_3 and s_4 are from district d_2. The initial matching is $\{(s_1, c_1), (s_2, c_2), (s_3, c_3), (s_4, c_3)\}$.

Given any set of contacts, district d_1 chooses students who have contracts with school c_1 first and then chooses from the remaining students who have contracts with school c_2. For school c_1, the district prioritizes students in the order $s_3 \succ s_4 \succ s_1 \succ s_2$ and chooses one applicant if there is any. For school c_2, the district prioritizes students according to the order $s_1 \succ s_2 \succ s_3 \succ s_4$ and chooses as many applicants as possible without going over the school's capacity while ignoring the contracts of the students who have already been accepted at school c_1. Likewise, district d_2 prioritizes students according to the order $s_3 \succ s_4 \succ s_1 \succ s_2$ and chooses as many applicants as possible without going over the capacity of school c_3. These admissions rules are feasible and acceptant, and they have completions that satisfy substitutability and LAD. In addition, student preferences are given by the following table,

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}
c_1	c_3	c_1	c_2
c_2	c_1	c_2	c_1
c_3	c_2	c_3	c_3

which means that, for instance, student s_1 prefers c_1 to c_2 to c_3.

In this problem, SPDA runs as follows. At the first step, student s_1 proposes to district d_1 with contract (s_1, c_1), student s_2 proposes to district d_2 with contract (s_2, c_3), student s_3 proposes to district d_1 with contract (s_3, c_1), and student s_4 proposes to district d_1 with contract (s_4, c_2). District d_1 first considers contracts associated with school c_1, (s_1, c_1) and (s_3, c_1), and tentatively accepts (s_3, c_1) while rejecting (s_1, c_1) because student s_3 has a higher priority than student s_1 at school c_1. Then district d_1 considers contracts of the remaining students associated with school c_2. In this case, there is only one such contract, (s_4, c_2), which is tentatively accepted. District d_2 considers contract (s_2, c_3) and tentatively accepts it. The tentative matching is $\{(s_2, c_3), (s_3, c_1), (s_4, c_2)\}$. Since there is a rejection, the algorithm proceeds to the next step.

At the second step, student s_1 proposes to district d_1 with contract (s_1, c_2). District d_1 first considers contract (s_3, c_1) and tentatively accepts it. Then district d_1 considers contracts (s_1, c_2) and (s_4, c_2) and tentatively accepts them both. District d_2 does not have any new contracts, so tentatively accepts (s_2, c_3). Since there is no rejection, the algorithm stops. The outcome of SPDA is $\{(s_1, c_2), (s_2, c_3), (s_3, c_1), (s_4, c_2)\}$. □

3. Results

In this section, we formalize four policy goals and characterize conditions under which SPDA satisfies them.[20]

3.1. Individual Rationality. In our context, individual rationality requires that every student is matched with a weakly more preferred school than her initial school. As a result, SPDA does not necessarily satisfy individual rationality even though each student is either unmatched or matched with a school that is more preferred than being unmatched.

If individual rationality is violated so that some students prefer their initial schools to the outcome of SPDA, then there may be public opposition that harm interdistrict school

[20]Our focus on SPDA is motivated by the fact that, under mild conditions, SPDA is a uniquely appealing mechanism. More specifically, Hatfield et al. (2021) present conditions under which the SPDA is the unique stable and strategy-proof mechanism, namely "observable substitutability," "observable size monotonicity," and "non-manipulability via contractual terms." It is a matter of verification that their conditions are satisfied in our setting.

choice efforts. For this reason, individual rationality is a desirable property for policymakers.[21] The following condition proves to play a crucial role for achieving this property.

Definition 1. *A district admissions rule* Ch_d ***respects the initial matching*** *if, for any student* s *whose initial school* c *is in district* d *and matching* X *that is feasible for students,*

$$(s, d, c) \in X \implies (s, d, c) \in Ch_d(X).$$

When a district's admissions rule respects the initial matching, it has to admit those contracts associated with itself in which students apply to their initial schools from every matching that is feasible for students. The following result shows that this is exactly the condition for SPDA to satisfy individual rationality.

Theorem 1. *SPDA satisfies individual rationality if and only if each district's admissions rule respects the initial matching.*

The intuition for the "if" part of this theorem is simple. When district admissions rules respect the initial matching, no student is matched with a school which is strictly less preferred than her initial school under SPDA because she is guaranteed to be accepted by that school if she applies to it. For the "only if" part of the theorem, we construct a specific student preference profile such that SPDA assigns one student a strictly less preferred school than her initial school whenever there exists one district with an admissions rule that does not respect the initial matching.

3.1.1. *District Admissions Rules Satisfying the Assumptions in Theorem 1.* We modify the district admissions rule construction in Section 2.2.1 and further specify each school's admissions rule. Each school has a responsive admissions rule. If a student is initially matched with a school, then her contract with this school is ranked higher than contracts of students who are not initially matched with the school. We call this admission rule Ch^i_d. As before, district admissions rule Ch^i_d is feasible and acceptant, and it has a completion that satisfies substitutability and LAD. The proof of the following claim is relegated to Appendix C.

Claim 4. *District admissions rule* Ch^i_d *respects the initial matching.*

3.1.2. *An example.* In the next example, we illustrate SPDA with district admissions rules that respect the initial matching.

[21]Pereyra (2013) considers dynamic matching, in which his concept of individual rationality requires that each applicant (teacher) must receive an outcome that is at least as desirable for her as her outcome in the previous period. Our definition of individual rationality requires the outcome of any applicant (student) be at least as desirable for her as an exogenously given initial matching. Although there is no formal relationship between those concepts due to model differences, they share the idea of individual rationality as a requirement to guarantee the outcome be at least as desirable as a certain default matching.

Example 2. Consider the problem in Example 1. Recall that in this problem, the outcome of SPDA is $\{(s_1, c_2), (s_2, c_3), (s_3, c_1), (s_4, c_2)\}$. This matching is not individually rational because student s_1 prefers her initial school c_1 to school c_2 that she is matched with. This observation is consistent with Theorem 1 because the admissions rule of district d_1 does not respect the initial matching. In particular, $Ch_{d_1}(\{(s_1, c_1), (s_3, c_1)\}) = \{(s_3, c_1)\}$, so student s_1 is rejected from a matching that is feasible for students and includes the contract with her initial school.

Now modify the priority ranking of district d_1 at school c_1 so that $s_1 \succ s_2 \succ s_3 \succ s_4$ but, otherwise, keep the construction of the district admissions rules and student preferences the same as before. With this change, district admissions rules respect the initial matching because each student is accepted when she applies to the district with her initial school. □

In some school districts, each student gets a priority at her neighborhood school. In the absence of other types of priorities, neighborhood priority guarantees that SPDA satisfies individual rationality if the initial school for each student is her neighborhood school and district choice rules are constructed as in Example 1.

3.2. Improving Student Welfare for Districts with Intradistrict School Choice. In Section 3.1, we studied when SPDA satisfies individual rationality, which requires that, under interdistrict school choice, every student is matched with a school that is weakly more preferred than her initial school. In this section, we consider an alternative setting where each district uses SPDA to assign its students to schools when there is no interdistrict school choice. In other words, the status quo is SPDA when there is only intradistrict school choice. More explicitly, each student ranks schools in their home districts (or contracts associated with their home districts) and SPDA is used between a district and students from that district. Note that we assume each student's ranking over contracts associated with the home district is the same as the relative ranking in the original preferences. Importantly, in this setting, we compare SPDA outcomes in interdistrict and intradistrict school choice. In such a setting, we characterize district admissions rules which guarantee that no student is hurt from interdistrict school choice.

The next property of district admissions rules plays a crucial role to achieve this policy.

Definition 2. *A district admissions rule Ch_d* ***favors own students*** *if for any matching X that is feasible for students,*

$$Ch_d(X) \supseteq Ch_d(\{x \in X | d(s(x)) = d\}).$$

When a district admissions rule favors own students, any contract that is chosen from a set of contracts associated with students from a district is also chosen from a superset that includes additional contracts associated with students from the other districts. Roughly, this condition requires that, under interdistrict school choice, a district prioritizes its own students that it used to admit over students from the other districts (even though an out-of-district student can still be admitted when a student from the district is rejected).

The following result shows that this is exactly the condition which guarantees that interdistrict school choice weakly improves the outcome for every student.

Theorem 2. *Every student weakly prefers the SPDA outcome under interdistrict school choice to the SPDA outcome under intradistrict school choice for all student preferences if and only if each district's admissions rule favors own students.*

In the proof, we show that in the intradistrict school choice the SPDA outcome can alternatively be produced by an interdistrict school choice model where students rank contracts with all districts and districts have modified admissions rules: For any set of contracts X, each district d chooses the following contracts: $Ch_d(\{x \in X | d(s(x)) = d\})$. Since the original district admissions rules favor own students, the chosen set under the modified admissions rule is a subset of $Ch_d(X)$ when X is feasible for students. Then the conclusion that students receive weakly more preferred outcomes in interdistrict school choice than in intradistrict school choice follows from a comparative statics property of SPDA that is shown to hold in the proof of this theorem.[22] To show the "only if" part, when there exists a district admissions rule that fails to favor own students, we construct student preferences such that interdistrict school choice makes at least one student strictly worse off than intradistrict school choice.

3.2.1. *District Admissions Rules Satisfying the Assumptions in Theorem 2.* Consider the district admissions rule construction in Section 2.2.1. In this example, let each school use a priority ranking in such a way that all contracts of students from district d are ranked higher than the other contracts. We call this admission rule Ch_d^w. The proof of the following claim is relegated to Appendix C.

Claim 5. *District admissions rule Ch_d^w favors own students.*

3.2.2. *An example.* Next we provide an example of district integration when admissions rules favor own students.

Example 3. Consider the integration problem in Example 1 again. Now, each district prioritizes its own students. In particular, the priority ranking of students for district d_1 is $s_1 \succ s_2 \succ s_3 \succ s_4$ and the priority ranking of students for district d_2 is $s_3 \succ s_4 \succ s_1 \succ s_2$. From any given set of contracts, each district chooses as many students as possible so

[22]We cannot use the comparative statics result of Yenmez (2018) because in our setting $Ch_d(X) \supseteq Ch'_d(X)$ only when X is feasible for students, whereas Yenmez (2018) requires this property for all sets of contracts X.

that the number of students in each school is at most its capacity. Then for instance, $Ch^w_{d_1}(\{(s_1, c_1), (s_3, c_1), (s_4, c_2)\}) = \{(s_1, c_1), (s_4, c_2)\}$. Student s_3 is rejected because school c_1 has capacity one and student s_1 has a higher priority than student s_3 at district d_1.

District admissions rules favor own students as each district prioritizes its students. In particular, when a contract of one of its students is chosen from a feasible matching, the district also chooses the same contract from the subset of contracts that are associated with students who are from the district. Then the implication of Theorem 2 is that each student is weakly better off from integration under SPDA. Let us verify this.

At Step 1, student s_1 proposes to district d_1 with contract (s_1, c_1), student s_2 proposes to district d_2 with contract (s_2, c_3), student s_3 proposes to district d_1 with contract (s_3, c_1), and student s_4 applies to district d_1 with contract (s_4, c_2). District d_1 accepts contracts (s_1, c_1) and (s_4, c_2), and rejects contract (s_3, c_1). District d_2 accepts (s_2, c_3). The tentative matching at the end of Step 1 is $\{(s_1, c_1), (s_2, c_3), (s_4, c_2)\}$. Student s_3 is unmatched.

At Step 2, student s_3 proposes to district d_1 with contract (s_3, c_2). District d_1 considers the set of contracts $\{(s_1, c_1), (s_3, c_2), (s_4, c_2)\}$. Since the capacity of school c_2 is two, district d_1 accepts all contracts in this set. Since there is no rejection, SPDA terminates. The outcome is $\{(s_1, c_1), (s_2, c_3), (s_3, c_2), (s_4, c_2)\}$.

Compared to their initial matching found in Example 1, students s_2, s_3, and s_4 all get better schools while student s_1 gets the same school. This corroborates the implications of Theorems 2 and 1.

3.3. Balanced Exchange. For interdistrict school choice, maintaining a balance of students incoming from and outgoing to other districts is important. To formalize this idea, we say that a mechanism satisfies the ***balanced-exchange*** policy if the number of students that a district gets from the other districts and the number of students that the district sends to the others are the same for every district and for every profile of student preferences. Since district choice rules are acceptant and students prefer every school to the outside option of being unmatched, every student is matched with a school under SPDA. Therefore, for SPDA, this policy is equivalent to the requirement that the number of students assigned to a district must be equal to the number of students from that district.

The balanced-exchange policy is important because the funding that a district gets depends on the number of students it serves.[23] Therefore, an interdistrict school choice program may not be sustainable if SPDA does not satisfy the balanced-exchange policy. For achieving this policy goal, the following condition on admissions rules proves important.

Definition 3. *A matching X is **rationed** if, for every district d, it does not assign strictly more students to the district than the number of students whose home district is d. A district admissions rule is **rationed** if it chooses a rationed matching from any matching that is feasible for students.*

When a district admissions rule is rationed, the district does not accept strictly more students than the number of students from the district at any matching that is feasible for students. The result below establishes that this property is exactly the condition to guarantee that SPDA satisfies the balanced-exchange policy.

Theorem 3. *SPDA satisfies the balanced-exchange policy if and only if each district's admissions rule is rationed.*

To obtain the intuition for this result, consider a student. Acceptance requires that a district can reject all contracts of this student only when the number of students assigned to the district is at least as large as the number of students from that district. As a result, all students are guaranteed to be matched. In addition, when district admissions rules are rationed, a district cannot accept more students than the number of students from the district. These two facts together imply that the number of students assigned to a district in SPDA is equal to the number of students from that district. Therefore, SPDA satisfies the balanced-exchange policy when each district's admissions rule is rationed. Conversely, when there exists one district with an admissions rule that fails to be rationed, then we can construct student preferences such that this district is matched with strictly more students than the number of students from the district in SPDA, which means that the outcome does not satisfy the balanced-exchange policy.

3.3.1. *District Admissions Rules Satisfying the Assumptions in Theorem 3.* We modify the district admissions rule construction in Section 2.2.1. Each school has a ranking of contracts associated with itself. When it is the turn of a school, it accepts contracts that have the highest rank until the capacity of the school is full, or the number of contracts chosen by

[23]For the United States, Levin et al. (2019) describes and discusses "weighted student formula (WSF) funding," under which each school receives funding based on student enrollment and each student's individual characteristics. According to this report, many districts including San Francisco and New York uses WSF. In the United Kingdom, Department for Education webpage (`https://www.gov.uk/government/publications/guide-to-national-funding-formula/guide-to-national-funding-formula`, last accessed on August 9, 2021.) states that the schools national funding formula (NFF) calculates an allocation for each school by using the school's pupil numbers and characteristics from the previous October schools census.

the district is k_d, or there are no more contracts left. The remaining contracts of a chosen student are removed. We call this admission rule Ch_d^b.

District admissions rule Ch_d^b is feasible because no school admits more students than its capacity and no student is admitted to more than one school. Proofs of the following claims are relegated to Appendix C.

Claim 6. *District admissions rule Ch_d^b is acceptant.*

Claim 7. *District admissions rule Ch_d^b has a completion that satisfies substitutability and LAD.*

Furthermore, by construction, district admissions rule Ch_d^b never chooses more than k_d students. Therefore, it is also rationed.

3.3.2. *An example.* Now we illustrate SPDA when district admissions rules are rationed.

Example 4. Consider the problem in Example 1. Recall that in this problem, the SPDA outcome is $\{(s_1, c_2), (s_2, c_3), (s_3, c_1), (s_4, c_2)\}$. Since there are three students matched with district d_1 and there are only two students from that district, SPDA does not satisfy the balanced-exchange policy. This is consistent with Theorem 3 because the admissions rule of district d_1 is not rationed. In particular, $Ch_{d_1}(\{(s_1, c_2), (s_3, c_1), (s_4, c_2)\}) = \{(s_1, c_2), (s_3, c_1), (s_4, c_2)\}$, so district d_1 accepts strictly more students than the number of students from there given a matching that is feasible for students.

Suppose that we modify the admissions rule of district d_1 as follows. If the district chooses a contract associated with school c_1, then at most one contract associated with school c_2 is chosen. Therefore, the district never chooses more than two contracts, which is the number of students from there. Therefore, the updated admissions rule is rationed. $\square$

An implication of Theorems 1 and 3 is that SPDA satisfies individual rationality and the balanced-exchange policy if and only if each district's admissions rule respects the initial matching and is rationed.

3.4. Diversity. The fourth policy goal we consider is that of diversity. More specifically, we are interested in how to ensure that there is enough diversity across districts so that the student composition in terms of demographics does not vary too much from district to district.

We are mainly motivated by a program that is used in the state of Minnesota.[24] State law in Minnesota identifies racially isolated (relative to one of their neighbors) school

[24]Although we are not aware of any other explicit examples of "interdistrict" requirements for diversity, there are many "intradistrict" examples. For instance, New York City's "Educational Option" (EdOpt) and Jefferson County School District have plans that require schools to allocate students from different socioeconomic background within some percentages. See Ehlers et al. (2014) for more information.

districts and requires them to be in the *Achievement and Integration* (*AI*) *Program*. The goal is to increase the racial parity between neighboring school districts. We first introduce a diversity policy in the spirit of this program: Given a constant $\alpha \in [0,1]$, we say that a mechanism satisfies the ***α-diversity policy*** if for all preferences, districts d and d', and type t, the difference between the ratios of type-t students in districts d and d' is not more than α. We interpret α to be the maximum ratio difference tolerated under the diversity policy; for instance, $\alpha = 0.2$ for Minnesota.

We study admissions rules such that SPDA satisfies the α-diversity policy when there is interdistrict school choice. Since this policy restricts the number of students across districts, a natural starting point is to have type-specific ceilings at the district level. However, it turns out that type-specific ceilings at the district level may yield district admissions rules resulting in no stable matchings (see Theorem 5 in Appendix A).

Since there is an incompatibility between district-level type-specific ceilings and the existence of a stable matching, we impose type-specific ceilings at the school level as follows.

Definition 4. *A district admissions rule Ch_d has a* ***school-level type-specific ceiling*** *of q_c^t at school c for type-t students if the number of type-t students admitted cannot exceed this ceiling. More formally, for any matching X that is feasible for students,*

$$|\{x \in Ch_d(X) | \tau(s(x)) = t, c\,(x) = c\}| \leq q_c^t.$$

Note that district admissions rules typically violate acceptance once school-level type-specific ceilings are imposed. This is because a student can be rejected from a set that is feasible for students even when the number of applicants to each school is smaller than its capacity and the number of applicants to the district is smaller than the number of students from that district. Given this, we define a weaker version of the acceptance assumption as follows.

Definition 5. *A district admissions rule Ch_d that has school-level type-specific ceilings is* ***weakly acceptant*** *if, for any contract x associated with a type-t student and district d and matching X that is feasible for students, if x is rejected from X, then at $Ch_d(X)$,*

- *the number of students assigned to school $c(x)$ is equal to $q_{c(x)}$, or*
- *the number of students assigned to district d is at least k_d, or*
- *the number of type-t students assigned to school $c(x)$ is at least q_c^t.*

In other words, a student can be rejected from a set that is feasible for students only when one of these three conditions is satisfied.

In SPDA, a student may be left unassigned due to school-level type-specific ceilings even when district admissions rules are weakly acceptant. To make sure that every student is matched, we make the following assumption.

Definition 6. *A profile of district admissions rules* $(Ch_d)_{d\in\mathcal{D}}$ ***accommodates unmatched students*** *if for any student* s *and feasible matching* X *in which student* s *is unmatched, there exists* $x = (s, d, c) \in \mathcal{X}$ *such that* $x \in Ch_d(X \cup \{x\})$.

When a profile of district admissions rules accommodates unmatched students, for any feasible matching in which a student is unmatched, there exists a school such that the district associated with the school would admit that student if she applies to that school. For example, when each admissions rule respects the initial matching, the profile of district admissions rules accommodates unmatched students because an unmatched student's application to her initial school is always accepted. When a profile of district admissions rules accommodates unmatched students, every student is matched to a school in SPDA (Lemma 1).

In general, accommodation of unmatched students may be in conflict with type-specific ceilings because there may not be enough space for a student type when ceilings are small for this type. To avoid this, we assume that type-specific ceilings are high enough so that $(Ch_d)_{d\in\mathcal{D}}$ accommodates unmatched students.[25]

Our assumptions on district admissions rules allow us to control the distribution of the SPDA outcome. In particular, the SPDA outcome satisfies the following conditions: (i) $\sum_t \xi_d^t(X) = k_d$ for all $d \in \mathcal{D}$, (ii) $\sum_{c\in\mathcal{C}} \xi_c^t(X) = k^t$ for all $t \in \mathcal{T}$, (iii) $\sum_{t\in\mathcal{T}} \xi_c^t(X) \leq q_c$ for all $c \in \mathcal{C}$, and (iv) $\xi_c^t(X) \leq q_c^t$ for all $t \in \mathcal{T}$ and $c \in \mathcal{C}$. We call any matching X satisfying these conditions **legitimate**.

In this framework, type-t ceilings of schools in district d may result in a floor of another type t' in this district in the sense that the number of type-t' students in the district should be at least a certain number. Moreover, this may further impose a ceiling for type t' in another district d'. To see this, suppose, for example, that (i) there are two districts d and d', (ii) in each district, there is one school and 100 students, (iii) 100 students are of type t and 100 students are of another type t', and (iv) each school has a type-t ceiling of 60 and a type-t' ceiling of 70. In a legitimate matching, each district needs to have at least 40 type-t' students (because, otherwise, the number of type-t students in that district would have to be more than 60). Moreover, this would mean that there cannot be more than 60 type-t' students in any district (because, otherwise, there would need to be more than 40 type-t' students in the other district, contradicting the floor we just calculated). Hence, in this example, in effect we have a floor of 40 and a (further restricted) ceiling of 60 for type-t' students for each district.

[25]For instance, ignoring integer problems, $q_d^t \geq k_d \frac{k^t}{\sum_{t'\in\mathcal{T}} k^{t'}}$ for all t, d, would make ceilings compatible with this property as it would be possible to assign the same percentage of students of each type to all districts.

Faced with this complication, our approach is to find the tightest lower and upper bounds induced by these constraints. For this purpose, a certain optimization problem proves useful. More specifically, consider a linear-programming problem where for each type t and district d, we seek the minimum and maximum values of $\sum_{c:d(c)=d} y_c^t$ subject to (i) $\sum_{t'\in\mathcal{T}} \sum_{c:d(c)=d'} y_c^{t'} = k_{d'}$ for all $d' \in \mathcal{D}$, (ii) $\sum_{c\in\mathcal{C}} y_c^{t'} = k^{t'}$ for all $t' \in \mathcal{T}$, (iii) $\sum_{t'\in\mathcal{T}} y_c^{t'} \leq q_c$ for all $c \in \mathcal{C}$, and (vi) $y_c^{t'} \leq q_c^{t'}$ for all $t' \in \mathcal{T}$ and $c \in \mathcal{C}$. Let $\hat{p}_d^t$ and $\hat{q}_d^t$ be the solutions to the minimization and maximization problems, respectively.

Both of these optimization problems belong to a special class of linear-programming problems called a minimum-cost flow problem, and many computationally efficient algorithms to solve it are known in the literature.[26] A straightforward but important observation is that $\hat{p}_d^t$ (resp. $\hat{q}_d^t$) is exactly the lowest (resp. highest) number of type-t students who can be matched to district d in a legitimate matching (Lemma 2). Given this observation, we call $\hat{p}_d^t$ the ***implied floor*** and $\hat{q}_d^t$ the ***implied ceiling***.

Now we are ready to state the main result of this section.

Theorem 4. *Suppose that each district admissions rule has school-level type-specific ceilings and is rationed and weakly acceptant. Moreover, suppose that the profile of district admissions rule accommodates unmatched students. Then, SPDA satisfies the α-diversity policy if and only if $\hat{q}_d^t/k_d - \hat{p}_{d'}^t/k_{d'} \leq \alpha$ for every type t and districts d, d' such that $d \neq d'$.*

The proof of this theorem, given in Appendix B, is based on a number of steps. First, as mentioned above, we note that $\hat{p}_d^t$ and $\hat{q}_d^t$ are the lower and upper bounds, respectively, of the number of type-t students who can be matched with district d in any legitimate matching. This observation immediately establishes the "if" part of the theorem. Then, we further show that the implied floors and ceilings can be achieved simultaneously in the sense that, for any pair of districts d and d' with $d \neq d'$, there exists a legitimate matching that assigns exactly $\hat{q}_d^t$ type-t students to district d and exactly $\hat{p}_{d'}^t$ type-t students to district d' (Lemma 3). In other words, we establish that the implied ceiling and floor are achieved in two different districts, and they are achieved at *one* legitimate matching simultaneously. We complete the proof of the theorem by constructing student preferences such that the outcome of SPDA achieves these bounds. In Appendix D, we provide an example that illustrates Theorem 4. In the next section, we provide a fairly general class of district admissions rules that satisfies our assumptions in this result.

[26] To see that our problem is a minimum-cost flow problem, note that we can take $(k_d)_{d\in\mathcal{D}}$ as the "supply," $(k^t)_{t\in\mathcal{T}}$ as the "demand," $(q_d^t)_{d\in\mathcal{D},t\in\mathcal{T}}$ as the "arc capacity bounds," and the objective functions for $\hat{p}_d^t$ and $\hat{q}_d^t$ to be $\min y_d^t$ and $\min -y_d^t$, respectively. These problems have an "integrality property" so that if the supply, demand, and bounds are integers, then all the solutions are integers as well. As already mentioned, many algorithms have been proposed to solve different objective functions for these problems. For instance, the capacity scaling algorithm of Edmonds and Karp (1972) gives the solutions in polynomial time. For more information, see Chapter 10 of Ahuja (2017). We are grateful to Fatma Kilinc-Karzan for helpful discussions.

3.4.1. *District Admissions Rules Satisfying the Assumptions in Theorem 4.* A profile of district admissions rules can accommodate unmatched students by *reserving* seats for different types of students:

Definition 7. *Let c be a school in district d. A district admissions rule Ch_d has a* **reserve** *of r_c^t for type-t students at school c if, for any feasible matching X that does not have any contract associated with type-t student s, if $|\{x \in X_c | \tau(s(x)) = t\}| < r_c^t$, then $x = (s, d, c)$ satisfies $x \in Ch_d(X \cup \{x\})$.*

A reserve for a student type at a school c guarantees space for this type at school c. Therefore, when a student is unmatched at a feasible matching and the reserve for her type is not yet filled at a school, the district will accept this student at that school if she applies to it. Proof of the following claim is relegated to Appendix C.

Claim 8. *Suppose that districts have admissions rules with reserves such that $\sum_c r_c^t = k^t$ for every type t. Then the profile of district admissions rules accommodates unmatched students.*

A district can have type-specific reserves at its schools in different ways. In the rest of this Section, we use school admissions rules with reserves introduced by Hafalir et al. (2013) to construct a fairly general example in which a district has schools with type-specific reserves. Let r_c^t be the number of seats reserved by school c for type-t students. Suppose that the type-specific ceilings for schools are given and that they satisfy the assumptions in Section 3.4. Assume that, for every district d, $\sum_{c:d(c)=d} \sum_t r_c^t = k_d$, $\sum_c r_c^t = k^t$ and, for every type t and school c, $r_c^t \leq q_c^t$. Furthermore, assume that $\sum_t r_c^t \leq q_c$ for every school c.

Consider the following district admissions rule for district d. Schools are ordered as $c_1, c_2, \ldots, c_n$. Each school has a ranking over contracts associated with it and a linear order over student types. First, all schools choose contracts for their reserved seats according to the order $c_1, c_2, \ldots, c_n$. When it is the turn of school c_i, all contracts associated with students whose contracts were previously chosen are removed. School c_i chooses contracts for its reserved seats so that, for every type, either reserved seats are filled or there are no more contracts associated with students of that type remaining. Then all schools choose contracts for their empty seats following the given order. When it is the turn of school c_i, all contracts of previously chosen students are removed. School c_i chooses from the remaining contracts in order. When a contract of a type-t student is considered, this contract is chosen unless the school's capacity is filled or its type-t ceiling is filled or the district has k_d contracts. Denote this district admissions rule by Ch_d^d.

District admissions rule Ch_d^d is feasible because a student cannot have more than one contract and a school cannot have more contracts than its capacity at any chosen set of contracts. It is also weakly acceptant and rationed by construction. Furthermore, for every type t and school c, the district cannot admit more than q_c^t type-t students at c, so it has a school-level type-specific ceiling of q_c^t for type-t students and school c. A proof of the following claim is relegated to Appendix C.

Claim 9. *District admissions rule Ch_d^d has a completion that satisfies substitutability and LAD.*

The analysis in this section characterizes conditions under which different policy goals are achieved under SPDA. One of the facts worth mentioning in this context is that achieving multiple policies can be overly demanding. To see this point, we note that individual rationality and α-diversity policy are often incompatible with one another. For example, consider a problem such that each student's most preferred school is her initial school and a constant α such that the initial matching does not satisfy the α-diversity policy. Indeed, in this case, no mechanism can simultaneously satisfy individual rationality and the α-diversity policy because the initial matching is the unique individually rational matching, but it fails the α-diversity policy.

4. Conclusion

Despite increasing interest in interdistrict school choice in the US, the scope of matching theory has been limited to intradistrict choice. In this paper, we proposed a new framework to study interdistrict school choice that allows for interdistrict admissions from stability perspective. For stable mechanisms, we characterized conditions on district admissions rules that achieve a variety of important policy goals, such as student diversity across districts. Overall, our analysis suggests that interdistrict school choice can help achieve desirable policy goals such as student diversity, but only with an appropriate design of constraints, admissions rules, and assignment mechanisms.

We regard this paper as a first step toward formal analysis of interdistrict school choice based on tools of market design. As such, we envision a variety of directions for future research. For example, it may be interesting to study cases in which the conditions for our results are violated. Although we already know the policy goals are not guaranteed to be satisfied for our stability results (our results provide necessary and sufficient conditions), the seriousness of the failure of the policy goals studied in the present paper is an open question. Quantitative measures or an approximation argument like those used in "large matching market" studies (e.g., Roth and Peranson (1999), Kojima and Pathak (2009), Kojima et al. (2013), Azevedo and Leshno (2016), and Ashlagi et al. (2014)) may prove useful, although this is speculative at this point and beyond the scope of the present paper.

We studied policy goals that we regarded as among the most important ones, but they are far from being exhaustive. Other important policy goals may include a diversity policy requiring certain proportions of different student types in each district (see Nguyen and Vohra (2017) for a related policy at the level of schools), as well as a balanced exchange policy requiring a certain bound on the difference in the numbers of students received from and sent to other districts (see Dur and Ünver (2019) for a related policy at the level of schools). Given that the existing literature has not studied interdistrict school choice, we envision that many policy goals await to be studied within our framework. Furthermore, we suspect that it may be possible to define a weaker notion than stability and seek mechanisms that satisfy the weakened condition while also satisfying constraints studied in this paper. Such an analysis would be substantially different from what we do in the present paper, however, and we submit it as a topic for future research.

While our paper is primarily theoretical and aimed at proposing a general framework to study interdistrict school choice, the main motivation comes from applications to actual programs such as Minnesota's AI program. Given this motivation, it would be interesting to study interdistrict school choice empirically. For instance, evaluating how well the existing programs are doing in terms of balanced exchange, student welfare, and diversity, and how much improvement could be made by a conscious design based on theories such as the ones suggested in the present paper, are important questions left for future work. In addition, implementation of our designs in practice would be interesting. We are only beginning to learn about the interdistrict school choice problem, and thus we expect that these and other questions could be answered as more researchers analyze it.

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Appendix A. An Additional Result

In this section, we show the incompatibility of type-specific ceilings at the district level with the existence of a stable matching.

Definition 8. *A district admissions rule Ch_d has a **district-level type-specific ceiling** of q_d^t for type-t students if the number of type-t students admitted from a matching that is feasible for students cannot exceed this ceiling. More formally, for any matching X that is feasible for students,*

$$|\{x \in Ch_d(X) | \tau(s(x)) = t\}| \leq q_d^t.$$

Note that, as in the case of school-level type-specific ceilings, district admissions rules do not necessarily satisfy acceptance once district-level type-specific ceilings are imposed. We define a weaker version of the acceptance assumption as follows.

Definition 9. *A district admissions rule Ch_d that has district-level type-specific ceilings is **d-weakly acceptant** if, for any contract x associated with a type-t student and district d and matching X that is feasible for students, if x is rejected from X, then at $Ch_d(X)$,*

- *the number of students assigned to school $c(x)$ is equal to $q_{c(x)}$, or*
- *the number of students assigned to district d is at least k_d, or*
- *the number of type-t students assigned to district d is at least q_d^t.*

This admissions rule property states that a student can be rejected only when one of these three conditions is satisfied.

We establish that in an interdistrict school choice problem in which district admissions rules have district-level type-specific ceilings that also satisfy some other desired properties, there may exist no stable matching.

Theorem 5. *There exist districts, schools, students, and their types such that for every admissions rule of a district with district-level type-specific ceilings that satisfies d-weak acceptance and IRC, there exist admissions rules for the other districts that satisfy substitutability and IRC and student preferences such that no stable matching exists.*

To show this result, we construct an environment such that a district admissions rule with the desired properties cannot satisfy *weak substitutability*, a necessary condition to guarantee the existence of a stable matching (Hatfield and Kojima, 2008).

Appendix B. Omitted Proofs of the Theorems

In this section, we provide the omitted proofs. Before proceeding with the proofs, some useful notation is in order. An admissions rule Ch_d satisfies ***path independence*** if for every $X, Y \subseteq \mathcal{X}$, $Ch_d(X \cup Y) = Ch_d(X \cup Ch_d(Y))$. Path independence states that a set can be divided into not-necessarily disjoint subsets and the admissions rule can be applied to the subsets in any order so that the chosen set of contracts is always the same. An admissions rule Ch_d satisfies ***the irrelevance of rejected contracts*** (IRC) if for every $X \subseteq \mathcal{X}$ and $x \notin Ch_d(X)$, $Ch_d(X \setminus \{x\}) = Ch_d(X)$. The irrelevance or rejected contracts states that a rejected contract can be removed from a set without changing the chosen set. Path independence is equivalent to substitutability and IRC (Aizerman and Malishevski, 1981).

Proof of Theorem 1. First, to show the "if" part, suppose that all district admissions rules respect the initial matching. In SPDA, each student s goes down in her preference order, and either SPDA ends before student s reaches her initial school (which is a preferred outcome over the initial school), or student s reaches her initial school. In the latter case, she is matched with her initial school because the district's admissions rule respects the initial matching and the district always considers a set of contracts that is feasible for students at any step of SPDA. From this step on, the district accepts this contract, so student s is matched with her initial school. Therefore, SPDA satisfies individual rationality.

To prove the "only if" part, suppose that there exists a district d with an admissions rule that fails to respect the initial matching. Hence, there exists a matching X, which is feasible for students, that includes $x = (s, d, c)$ where school c is the initial school of student s and $x \notin Ch_d(X)$. Now, consider student preferences such that every student associated with a contract in X_d prefers that contract the most and all other students prefer a contract associated with a different district the most. Then, at the first step of SPDA, district d considers matching X_d and tentatively accepts $Ch_d(X_d)$. Since $x \notin Ch_d(X_d)$, contract x is rejected at the first step. Therefore, student s is matched with a strictly less preferred school than her initial school, which implies that SPDA does not satisfy individual rationality. □

Proof of Theorem 2. Suppose that district admissions rules favor own students. Fix a student preference profile. Recall that under interdistrict school choice, students are assigned to schools by SPDA, where each student ranks all contracts associated with her and each district d has the admissions rule Ch_d. Under intradistrict school choice, students are assigned to schools by SPDA where students only rank the contracts associated with their home districts and each district d has the admissions rule Ch_d. We first show that the intradistrict SPDA outcome can be produced by SPDA when all districts participate simultaneously and students rank all contracts, including the ones associated with the other districts, by modifying admissions rules for the districts. Let $Ch'_d(X) \equiv Ch_d(\{x \in X | d(s(x)) = d\})$ be the modified admissions rule.

In SPDA, if district admissions rules have completions that satisfy path independence, then SPDA outcomes are the same under the completions and the original admissions rules because in SPDA a district always considers a set of contracts which is feasible for students. Furthermore, SPDA does not depend on the order of proposals when district admissions rules are path independent. As a result, SPDA does not depend on the order of proposals when district admissions rules have completions that satisfy path independence. Therefore, the intradistrict SPDA outcome can be produced by SPDA when all districts participate simultaneously and students rank all contracts including the ones associated with the other districts and each district d has the admissions rule Ch'_d. The reason behind this is that when each district d has admissions rule Ch'_d, a student is not admitted to a school district other than her home district. Furthermore, the set of chosen students under Ch'_d is the same as that under Ch_d for any set of contracts of the form $\{x \in X | d(s(x)) = d\}$ for any set X.

We next show that Ch'_d has a path-independent completion. By assumption, for every district d, there exists a path-independent completion $\widetilde{Ch}_d$ of Ch_d. Let $\widetilde{Ch}'_d(X) \equiv \widetilde{Ch}_d(\{x \in X | d(s(x)) = d\})$ for any set of contracts X. We show that $\widetilde{Ch}'_d$ is a path-independent completion of Ch'_d. To show that $\widetilde{Ch}'_d(X)$ is a completion, consider a set X such that $\widetilde{Ch}'_d(X)$ is feasible for students. Let $X^* \equiv \{x \in X | d(s(x)) = d\}$. Then we have the following:

$$\widetilde{Ch}'_d(X) = \widetilde{Ch}_d(X^*) = Ch_d(X^*) = Ch'_d(X),$$

where the first equality follows from the definition of $\widetilde{Ch}'_d$, the second equality follows from the fact that $\widetilde{Ch}_d$ is a completion of Ch_d and $\widetilde{Ch}_d(X^*) = \widetilde{Ch}'_d(X)$ is feasible for students, and the third equality follows from the definition of Ch'_d. Therefore, $\widetilde{Ch}'_d$ is a completion of Ch'_d.

To show that $\widetilde{Ch}'_d$ is path independent, consider two sets of contracts X and Y. Let $X^* \equiv \{x \in X | d(s(x)) = d\}$ and $Y^* \equiv \{x \in Y | d(s(x)) = d\}$. Then we have the following:

$$\begin{aligned}
\widetilde{Ch}'_d(X \cup \widetilde{Ch}'_d(Y)) &= \widetilde{Ch}'_d(X \cup \widetilde{Ch}_d(Y^*)) \\
&= \widetilde{Ch}_d(X^* \cup \widetilde{Ch}_d(Y^*)) \\
&= \widetilde{Ch}_d(X^* \cup Y^*) \\
&= \widetilde{Ch}'_d(X \cup Y),
\end{aligned}$$

where the first and second equalities follow from the definition of $\widetilde{Ch}'_d$, the third equality follows from path independence of $\widetilde{Ch}_d$, and the last equality follows from the definition of $\widetilde{Ch}'_d$. Therefore, $\widetilde{Ch}'_d$ is path independent.

Let Ch^*_d be defined as follows, for any set of contracts X,

$$Ch^*_d(X) = \widetilde{Ch}_d(X) \cup \widetilde{Ch}'_d(X).$$

We show that Ch^*_d is path independent by proving that it is substitutable and it satisfies the irrelevance of rejected contracts. To show substitutability, let $x \in X \subseteq Y \subseteq \mathcal{X}$ with $x \in Ch^*_d(Y)$. Then we have $x \in \widetilde{Ch}_d(Y)$ or $x \in \widetilde{Ch}'_d(Y)$ by the construction of Ch^*_d. Since $\widetilde{Ch}_d$ is substitutable, $x \in \widetilde{Ch}_d(Y)$ implies $x \in \widetilde{Ch}_d(X)$. Likewise, $x \in \widetilde{Ch}'_d(Y)$ implies $x \in \widetilde{Ch}'_d(X)$ because $\widetilde{Ch}'_d$ substitutable. We conclude that $x \in \widetilde{Ch}_d(X) \cup \widetilde{Ch}'_d(X) = Ch^*_d(X)$. Therefore, Ch^*_d is substitutable. To show the irrelevance of rejected contracts, let $X \subseteq \mathcal{X}$

and $x \notin Ch^*_d(X)$. By the construction of Ch^*_d, $x \notin \widetilde{Ch}_d(X)$ and $x \notin \widetilde{Ch}'_d(X)$. Since $\widetilde{Ch}_d(X)$ and $\widetilde{Ch}'_d(X)$ satisfy the irrelevance of rejected contracts, $\widetilde{Ch}_d(X \setminus \{x\}) = \widetilde{Ch}_d(X)$ and $\widetilde{Ch}'_d(X \setminus \{x\}) = \widetilde{Ch}'_d(X)$. Therefore, $Ch^*_d(X \setminus \{x\}) = Ch^*_d(X)$, which means that Ch^*_d satisfies the irrelevance of rejected contracts.

Now we show that the outcome of interdistrict SPDA under $(Ch_d)_{d\in\mathcal{D}}$ is the same as the outcome of the interdistrict SPDA under $(Ch^*_d)_{d\in\mathcal{D}}$. Let X be a feasible set of contracts and d be a district. Then both $\widetilde{Ch}_d(X)$ and $\widetilde{Ch}'_d(X)$ are feasible for students. Furthermore,

$$Ch^*_d(X) = \widetilde{Ch}_d(X) \cup \widetilde{Ch}'_d(X) = Ch_d(X) \cup Ch'_d(X) = Ch_d(X),$$

where the first equality follows from the definition of Ch^*_d, the second follows from the facts that $\widetilde{Ch}_d$ is a completion of Ch_d, $\widetilde{Ch}'_d$ is a completion of Ch'_d, and X is feasible for students, and the last equality follows from the fact that X is feasible for students and Ch_d favors own students. Since $Ch_d(X) = Ch^*_d(X)$ for any set of contracts X that is feasible for students and district d, we conclude that the outcome of interdistrict SPDA under $(Ch_d)_{d\in\mathcal{D}}$ is the same as the outcome of the interdistrict SPDA under $(Ch^*_d)_{d\in\mathcal{D}}$.

We conclude the proof of the first part of the theorem as follows: The interdistrict SPDA outcome under $(Ch_d)_{d\in\mathcal{D}}$ is the same as the interdistrict SPDA outcome under $(Ch^*_d)_{d\in\mathcal{D}}$. Similarly, since for each district d, $\widetilde{Ch}'_d$ is a completion of Ch'_d, the interdistrict SPDA outcome under $(Ch'_d)_{d\in\mathcal{D}}$ is the same as the interdistrict SPDA outcome under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$. Furthermore, for each district d both Ch^*_d and $\widetilde{Ch}'_d$ are path independent and, for any set of contracts X, $Ch^*_d(X) \supseteq \widetilde{Ch}'_d(X)$, which implies that each student weakly prefers the interdistrict SPDA outcome under $(Ch^*_d)_{d\in\mathcal{D}}$ to the interdistrict SPDA outcome under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$ by Corollary 1 of Chambers and Yenmez (2017). The conclusion follows because the outcome of interdistrict SPDA under $(Ch_d)_{d\in\mathcal{D}}$ is weakly more preferred by students to the outcome of intradistrict SPDA (which is the same as the interdistrict SPDA outcome under $(\widetilde{Ch}'_d)_{d\in\mathcal{D}}$).

To prove the second part of the theorem, we show that if at least one district's admissions rule fails to favor own students, then there exists a student preference profile such that not every student weakly prefers the interdistrict SPDA outcome to the intradistrict SPDA outcome. Suppose that for some district d, there exists a matching X, which is feasible for students, such that $Ch_d(X)$ is not a superset of $Ch_d(X^*)$, where $X^* \equiv \{x \in X | d(s(x)) = d\}$. Now, consider a matching Y where (i) for every district d', all students from district d' are matched with schools in district d', (ii) Y is feasible, and (iii) $Y_d \supseteq Ch_d(X^*)$. The existence of such a Y follows from the fact that $Ch_d(X^*)$ is feasible and $k_{d'} \leq \sum_{c:d(c)=d'} q_c$, for every district d' (that is, there are enough seats in district d' to match all students from district d'). Since Y is feasible, $|Y_d| = k_d$, and Ch_d is acceptant, $Ch_d(Y_d) = Y_d$. Note that since Ch_d is acceptant, students with contracts in $X^* \setminus Ch_d(X^*)$ are matched with different schools in Y, i.e., $Y_s \neq X^*_s$ because for any school c with a contract in $X^* \setminus Ch_d(X^*)$, we must have $X^*_c = Y_c$. To show this last claim, let $x = (s, d, c) \in X^* \setminus Ch_d(X^*)$. Since Ch_d is acceptant and X^* can have at most k_d contracts, school c must have q_c contracts in $Ch_d(X^*)$. Since $Y_d \supseteq X^*$ and Y_d is feasible, we must have $X^*_c = Y_c$.

Now consider the following student preferences. First we consider students from district d. Each student s who has a contract in X^* ranks X^*_s as her first choice. Note that doing so is well defined because X^* is feasible for students. Each student s who has a contract in $X^* \setminus Ch_d(X^*)$ ranks contract Y_s as her second choice. Note that, in this case, Y_s cannot be the same as X^*_s as we discussed in the previous paragraph. Each student s who does not have a contract in X^* ranks Y_s as her first choice. Next we consider students from the other districts. Each student s who has a contract in X ranks X_s as her first choice. Any other student ranks a contract not associated with district d as her first choice. Complete the rest of the student preferences arbitrarily.

Consider SPDA for district d in intradistrict school choice. Since Ch_d has a path-independent completion, the order of proposals does not change the outcome. At the first step, let students who have a contract in X^* propose. District d chooses $Ch_d(X^*)$ and rejects $X^* \setminus Ch_d(X^*)$. At the second step, the rejected students at the first step and students without a contract in X^* propose their associated contracts in Y. The set of proposals that the district considers is Y_d. Since $Ch_d(Y_d) = Y_d$, no contract is rejected and SPDA stops and returns Y_d. In particular, every student who has a contract in $Ch_d(X^*)$ has the corresponding contract at the outcome.

In interdistrict SPDA, at the first step, each student who has a contract in X proposes that contract and every other student proposes a contract associated with a district different from d. District d considers X_d and tentatively accepts $Ch_d(X_d)$. Because $Ch_d(X_d) = Ch_d(X) \not\supseteq Ch_d(X^*)$ by assumption, at least one student who has a contract in $Ch_d(X^*)$ is rejected. Therefore, this student gets a strictly less preferred contract under interdistrict school choice than intradistrict school choice. $\square$

Proof of Theorem 3. We first prove that if each district admissions rule is rationed, then SPDA satisfies the balanced-exchange policy. Let X be the matching produced by SPDA for a given preference profile.

We begin by showing that each student must be matched with a school in X. Suppose, for contradiction, that student s is unmatched. Since X is a stable matching, every contract $x = (s, d, c)$ associated with the student is rejected by the corresponding district, i.e., $x \notin Ch_d(X \cup \{x\})$. Otherwise, student s and district d would like to match with each other using contract x, contradicting the stability of matching X. Since $X \cup \{x\}$ is feasible for students, acceptance implies that, for each district d, either every school in the district is full or that the district has at least k_d students at matching X. Both of them imply that the district has at least k_d students in matching X since the sum of the school capacities in district d is at least k_d. But this is a contradiction to the assumption that student s is unmatched since the existence of an unmatched student implies that there is at least one district d such that the number of students in X_d is less than k_d. Therefore, all students are matched in X.

Because X is the outcome of SPDA, it is feasible for students. Therefore, because district admissions rules are rationed, the number of students in district d cannot be strictly more than k_d for any district d. Furthermore, since every student is matched, the number of students in district d must be exactly k_d (because, otherwise, at least one student would have been unmatched.) As a result, SPDA satisfies the balanced-exchange policy.

Next, we prove that if at least one district's admissions rule fails to be rationed, then there exists a student preference profile under which SPDA does not satisfy the balanced-exchange policy. Suppose that there exist a district d and a matching X, which is feasible for students, such that $|Ch_d(X)| > k_d$. Consider a feasible matching X' such that (i) all students are matched, (ii) $X'_d = Ch_d(X)$, and (iii) for every district $d' \neq d$, $|X'_{d'}| \leq k_{d'}$. The existence of such X' is guaranteed since every district has enough capacity to serve its students (i.e., for every district d', $\sum_{c:d(c)=d'} q_c \geq k_{d'}$), and $|Ch_d(X)| > k_d$. Now, consider any student preferences, where every student likes her contract in X' the most.

We show that SPDA stops in the first step. For district $d' \neq d$, $X'_{d'}$ is feasible and the number of students matched to d' at $X'_{d'}$ is weakly less than $k_{d'}$. Since $Ch_{d'}$ is acceptant, $Ch_{d'}(X'_{d'}) = X'_{d'}$. For district d, we need to show that $Ch_d(X'_d) = X'_d$, which is equivalent to $Ch_d(Ch_d(X)) = Ch_d(X)$. Let Ch'_d be a completion of Ch_d that satisfies path independence. Because X and $Ch_d(X)$ are feasible for students, $Ch'_d(X) = Ch_d(X)$ and $Ch'_d(Ch'_d(X)) = Ch_d(Ch_d(X))$. Furthermore, since Ch'_d is path independent, $Ch'_d(Ch'_d(X)) = Ch'_d(X)$, which implies $Ch_d(Ch_d(X)) = Ch_d(X)$. As a result, $Ch_d(X'_d) = X'_d$. Therefore, SPDA stops at the first step since no contract is rejected.

Since SPDA stops at the first step, the outcome is matching X'. But X' fails the balanced-exchange policy because $|X'_d| = |Ch_d(X)| > k_d$. □

Proof of Theorem 4.

Lemma 1. *If a profile of district admissions rules accommodates unmatched students, every student is matched to a school in SPDA.*

Proof of Lemma 1. Let X be the outcome of SPDA for some preference profile. Suppose, for contradiction, that student s is unmatched. Since X is a stable matching and student s prefers any contract $x = (s, d, c)$ to being unmatched, $x \notin Ch_d(X \cup \{x\})$. But this is a contradiction to the assumption that the profile of district admissions rules accommodates unmatched students. □

Lemma 2. *For each type t, district d, and legitimate matching X, we have $\hat{q}_d^t \geq \xi_d^t(X) \geq \hat{p}_d^t$. Moreover, for each type t and district d, there exist legitimate matchings X and X' such that $\xi_d^t(X) = \hat{p}_d^t$ and $\xi_d^t(X') = \hat{q}_d^t$.*

Proof of Lemma 2. Observe that for every legitimate matching X, the induced distribution satisfies the constraints of the linear program. Therefore, the first part follows from the definition of the implied floors and ceilings. For the second part, note that there exists a solution to the linear program such that the ceiling and the floor are attained. Furthermore, every solution $y = (y_c^t)_{c \in \mathcal{C}, t \in \mathcal{T}}$ of the linear program can be supported by a legitimate matching X such that $y_c^t = \xi_c^t(X)$ for every c and t. □

Lemma 3. *For each $t \in \mathcal{T}$ and $d, d' \in \mathcal{D}$ with $d \neq d'$, there exists a legitimate matching X such that $\xi_d^t(X) = \hat{q}_d^t$ and $\xi_{d'}^t(X) = \hat{p}_{d'}^t$.*

Proof of Lemma 3. Let $\hat{X}$ be a legitimate matching such that $\xi_d^t(\hat{X}) = \hat{q}_d^t$ and $\mathcal{M}_0$ be the set of all legitimate matchings. Let

$$\mathcal{M}_1 \equiv \{X \in \mathcal{M}_0 | \xi_{d'}^t(X) = \hat{p}_{d'}^t\}.$$

$\mathcal{M}_1$ is nonempty due to Lemma 2. Next, let

$$\mathcal{M}_2 \equiv \{X \in \mathcal{M}_1 | \sum_{\tilde{t},\tilde{c}} | \xi_{\tilde{c}}^{\tilde{t}}(X) - \xi_{\tilde{c}}^{\tilde{t}}(\hat{X}) | \leq \sum_{\tilde{t},\tilde{c}} | \xi_{\tilde{c}}^{\tilde{t}}(X') - \xi_{\tilde{c}}^{\tilde{t}}(\hat{X}) | \text{ for every } X' \in \mathcal{M}_1\}.$$

$\mathcal{M}_2$ is nonempty because $\mathcal{M}_1$ is a finite set. We will show that for any $X \in \mathcal{M}_2$, $\xi_d^t(X) = \xi_d^t(\hat{X}) = \hat{q}_d^t$.

To prove the above claim, assume for contradiction that there exists $X \in \mathcal{M}_2$ such that $\xi_d^t(X) \neq \xi_d^t(\hat{X})$. By Lemma 2, $\xi_d^t(X) \neq \xi_d^t(\hat{X})$ implies that $\xi_d^t(X) < \xi_d^t(\hat{X})$. Then there exists c with $d(c) = d$ such that $\xi_c^t(X) < \xi_c^t(\hat{X})$. Consider the following procedure.

Step 0: Initialize by setting $(t_1, c_1) := (t, c)$. Note that $\xi_{c_1}^{t_1}(X) < \xi_{c_1}^{t_1}(\hat{X})$ by definition of c.

Step $i \geq 1$: Given sequences of type-school pairs $((t_j, c_j))_{1 \leq j \leq i}$ and $((t_{j+1}, c_j^*))_{1 \leq j < i}$, proceed as follows. We begin with (t_i, c_i). Note that (by assumption for $i = 1$, and as shown later for $i \geq 2$), $\xi_{c_i}^{t_i}(X) < \xi_{c_i}^{t_i}(\hat{X})$. Denote $d_i = d(c_i)$. Now,

1. Suppose that there exists $i' < i$ such that either (i) $c_{i'}^* = c_i$ or (ii) $\xi_{c_i}^{t_i}(X) < q_{c_i}$ and $d(c_{i'}^*) = d(c_i)$. If such an index i' exists, then set $(t_{i+1}, c_i^*) := (t_{i'+1}, c_{i'}^*)$.
2. Suppose not. Then, if there exists $t' \in \mathcal{T}$ such that $\xi_{c_i}^{t'}(X) > \xi_{c_i}^{t'}(\hat{X})$, then set $(t_{i+1}, c_i^*) := (t', c_i)$.

(3) If not, then note that $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{c_i}(X) < q_{c_i}$.[27] Also note that there exists a type-school pair (t', c') with $c' \neq c_i$ such that $\xi^{t'}_{c'}(X) > \xi^{t'}_{c'}(\hat{X})$ and $d(c') = d_i$ because $\sum_{\tilde{c}:d(\tilde{c})=d_i,\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(X) = \sum_{\tilde{c}:d(\tilde{c})=d_i,\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(\hat{X}) = k_{d_i}$.

(a) If $t' = t_i$, then let $\bar{X}$ be a matching such that

$$\xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) = \begin{cases} \xi^{t_i}_{c_i}(X) + 1 & \text{for } (\tilde{t}, \tilde{c}) = (t_i, c_i), \\ \xi^{t'}_{c'}(X) - 1 & \text{for } (\tilde{t}, \tilde{c}) = (t_i, c'), \\ \xi^{\tilde{t}}_{\tilde{c}}(X) & \text{otherwise.} \end{cases}$$

Note that $\bar{X} \in \mathcal{M}_1$.[28] Also, by construction, $\sum_{\tilde{t},\tilde{c}} \mid \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) - \xi^{\tilde{t}}_{\tilde{c}}(\hat{X}) \mid = \sum_{\tilde{t},\tilde{c}} \mid \xi^{\tilde{t}}_{\tilde{c}}(X) - \xi^{\tilde{t}}_{\tilde{c}}(\hat{X}) \mid -2 < \sum_{\tilde{t},\tilde{c}} \mid \xi^{\tilde{t}}_{\tilde{c}}(X) - \xi^{\tilde{t}}_{\tilde{c}}(\hat{X}) \mid$, which contradicts the assumption that $X \in \mathcal{M}_2$.

(b) Therefore, suppose that $t' \neq t_i$ and set $(t_{i+1}, c^*_i) := (t', c')$.

(4) The pair (t_{i+1}, c^*_i) created above satisfies $\xi^{t_{i+1}}_{c^*_i}(X) > \xi^{t_{i+1}}_{c^*_i}(\hat{X})$, so there exists $c' \in \mathcal{C}$ such that $\xi^{t_{i+1}}_{c'}(X) < \xi^{t_{i+1}}_{c'}(\hat{X})$. Set $c_{i+1} = c'$. Note that $\xi^{t_{i+1}}_{c_{i+1}}(X) < \xi^{t_{i+1}}_{c_{i+1}}(\hat{X})$.

We follow the procedure above to define $(t_1, c_1), (t_2, c^*_1), (t_2, c_2), (t_3, c^*_2), (t_3, c_3)$, and so forth. Because $\mathcal{T}$ is a finite set, we have i and $j > i$ with $t_i = t_j$. Consider the smallest j with this property (note that given such j, i is uniquely identified). Now, let $\bar{X}$ be a matching such that

$$\xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) = \begin{cases} \xi^{t_k}_{c_k}(X) + 1 & \text{for } (\tilde{t}, \tilde{c}) = (t_k, c_k) \text{ for any } k \in \{i, i+1, \ldots, j-1\}, \\ \xi^{t_{k+1}}_{c^*_k}(X) - 1 & \text{for } (\tilde{t}, \tilde{c}) = (t_{k+1}, c^*_k) \text{ for any } k \in \{i, i+1, \ldots, j-1\}, \\ \xi^{\tilde{t}}_{\tilde{c}}(X) & \text{otherwise.} \end{cases}$$

We will show $\bar{X} \in \mathcal{M}_1$. To do so, by construction of $\bar{X}$, first note that $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) \leq \sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(X) + 1 \leq q_{\tilde{c}}$ for any $\tilde{c} \in \{c_i, \ldots, c_{j-1}\}$ such that $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(X) < q_{\tilde{c}}$. Next, by construction of $\bar{X}$, $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) = \sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(X) = q_{\tilde{c}}$ for every $\tilde{c} \in \{c_i, \ldots, c_{j-1}\}$ such that $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(X) = q_{\tilde{c}}$. Moreover, $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) \leq \sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(X) = q_{\tilde{c}}$ for every $\tilde{c} \in \{c^*_i, \ldots, c^*_{j-1}\}$. Finally, for every $\tilde{c} \in \mathcal{C} \setminus \{c_i, \ldots, c_{j-1}, c^*_i, \ldots, c^*_{j-1}\}$, $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) = \sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(X) \leq q_{\tilde{c}}$. Thus, all school capacities are satisfied by $\bar{X}$. Also by construction of $\bar{X}$, for each $\tilde{d} \in \mathcal{D}$, $\sum_{\tilde{c}:d(\tilde{c})=\tilde{d}} \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) = \sum_{\tilde{c}:d(\tilde{c})=\tilde{d}} \xi^{\tilde{t}}_{\tilde{c}}(X) = k_{\tilde{d}}$, so $\bar{X}$ is rationed. Furthermore, for every $\tilde{c} \in \mathcal{C}$

[27] A proof of this fact is as follows. By an earlier argument, $\xi^{t_i}_{c_i}(X) < \xi^{t_i}_{c_i}(\hat{X})$. Moreover, by assumption $\xi^{\tilde{t}}_{c_i}(X) \leq \xi^{\tilde{t}}_{c_i}(\hat{X})$ for every $\tilde{t} \in \mathcal{T}$. Therefore, $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{c_i}(X) < \sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{c_i}(\hat{X}) \leq q_{c_i}$.

[28] A proof of this fact is as follows. Because $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{c_i}(X) < q_{c_i}$, $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{c_i}(\bar{X}) = \sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{c_i}(X) + 1 \leq q_{c_i}$. For every $\tilde{c} \neq c_i$, $\sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) \leq \sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(X) \leq q_{\tilde{c}}$. Thus, all school capacities are satisfied. For all $\tilde{c}, \tilde{t}$, $\xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) \leq \max\{\xi^{\tilde{t}}_{\tilde{c}}(X), \xi^{\tilde{t}}_{\tilde{c}}(\hat{X})\} \leq q^{\tilde{t}}_{\tilde{c}}$ by construction, so all type-specific ceilings are satisfied. And $\sum_{\tilde{t}\in\mathcal{T},\tilde{c}\in\mathcal{C}} \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) = \sum_{\tilde{t}\in\mathcal{T}} \xi^{\tilde{t}}_{\tilde{c}}(X)$ by definition of $\bar{X}$, so $\bar{X}$ is a legitimate matching. Finally, $\xi^{\tilde{t}}_{\tilde{d}}(\bar{X}) = \xi^{\tilde{t}}_{\tilde{d}}(X)$ for every $\tilde{t}$ and $\tilde{d}$, so $\bar{X} \in \mathcal{M}_1$.

and $\tilde{t} \in \mathcal{T}$, $\xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) \leq \max\{\xi^{\tilde{t}}_{\tilde{c}}(X), \xi^{\tilde{t}}_{\tilde{c}}(\hat{X})\}$ by construction, so all type-specific ceilings are satisfied. Moreover, by construction of $\bar{X}$, for each $\tilde{t} \in \mathcal{T}$, either $\xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) = \xi^{\tilde{t}}_{\tilde{c}}(X)$ for every $\tilde{c} \in \mathcal{C}$ or there exists exactly one pair of schools $\tilde{c}'$ and $\tilde{c}''$ in $\mathcal{C}$ such that $\xi^{\tilde{t}}_{\tilde{c}'}(\bar{X}) = \xi^{\tilde{t}}_{\tilde{c}'}(\bar{X}) + 1$, $\xi^{\tilde{t}}_{\tilde{c}''}(\bar{X}) = \xi^{\tilde{t}}_{\tilde{c}''}(\bar{X}) - 1$, and $\xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) = \xi^{\tilde{t}}_{\tilde{c}}(X)$ for every $\tilde{c} \in \mathcal{C} \setminus \{\tilde{c}', \tilde{c}''\}$. Thus, $\tilde{t} \in \mathcal{T}$, $\sum_{\tilde{c}\in\mathcal{C}} \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) = \sum_{\tilde{c}\in\mathcal{C}} \xi^{\tilde{t}}_{\tilde{c}}(X)$ for every $\tilde{t} \in \mathcal{T}$. Therefore, $\bar{X}$ is legitimate.

By construction of $\bar{X}$, either $\xi^{t}_{d'}(\bar{X}) = \xi^{t}_{d'}(X)$ or $\xi^{t}_{d'}(\bar{X}) = \xi^{t}_{d'}(X) - 1$. Therefore, $\bar{X} \in \mathcal{M}_1$. Furthermore, $\sum_{\tilde{t},\tilde{c}} \mid \xi^{\tilde{t}}_{\tilde{c}}(\bar{X}) - \xi^{\tilde{t}}_{\tilde{c}}(\hat{X}) \mid < \sum_{\tilde{t},\tilde{c}} \mid \xi^{\tilde{t}}_{\tilde{c}}(X) - \xi^{\tilde{t}}_{\tilde{c}}(\hat{X}) \mid$, since while creating the $\xi^{\tilde{t}}_{\tilde{c}}(\bar{X})$ entries, we add 1 to some entries of X that satisfy $\xi^{\tilde{t}}_{\tilde{c}}(X) < \xi^{\tilde{t}}_{\tilde{c}}(\hat{X})$ and subtract 1 from some entries of X that satisfy $\xi^{\tilde{t}}_{\tilde{c}}(X) > \xi^{\tilde{t}}_{\tilde{c}}(\hat{X})$. These lead to a contradiction to the assumption that $X \in \mathcal{M}_2$, which completes the proof. □

Now we are ready to prove the theorem. The "if" part follows from Lemmas 1 and 2. Specifically, by Lemma 1, SPDA produces a legitimate matching. Therefore, by Lemma 2, we have $\hat{p}^t_d \leq \xi^t_d(X) \leq \hat{q}^t_d$ for every $t \in \mathcal{T}$ and $d \in \mathcal{D}$. For each school district d, hence, the maximum proportion of type-t students that can be admitted is $\hat{q}^t_d/k_d$ and the minimum proportion of type t students that can be admitted is $\hat{p}^t_d/k_d$. Therefore, the ratio difference of type-t students in any two districts is at most $\max\limits_{d\neq d'}\{\hat{q}^t_d/k_d - \hat{p}^t_{d'}/k_{d'}\}$. We conclude that the α-diversity policy is achieved when $\hat{q}^t_d/k_d - \hat{p}^t_{d'}/k_{d'} \leq \alpha$ for every t, d, and d' with $d \neq d'$.

The "only if" part of the theorem follows from Lemma 3. Suppose that $\hat{q}^t_d/k_d - \hat{p}^t_{d'}/k_{d'} > \alpha$ for some t, d, and d' with $d \neq d'$. From Lemma 3, we know the existence of a legitimate matching X such that $\xi^t_d(X) = \hat{q}^t_d$ and $\xi^t_{d'}(X) = \hat{p}^t_{d'}$. Consider a student preference profile where each student prefers her contract in X the most. Then, since the admissions rules are weakly acceptant, SPDA ends at the first step as all applications are accepted. Thus X is the outcome of SPDA and, therefore, the α-diversity policy is not satisfied. □

Proof of Theorem 5. To show the result, we first introduce the following weakening of the substitutability condition (Hatfield and Kojima, 2008). A district admissions rule Ch_d satisfies ***weak substitutability*** if, for every $x \in X \subseteq Y \subseteq \mathcal{X}$ with $x \in Ch_d(Y)$ and $|Y_s| \leq 1$ for each $s \in \mathcal{S}$, it must be that $x \in Ch_d(X)$.

Under weak substitutability, the following result is known (the statement is slightly modified for the present setting).

Theorem 6 (Hatfield and Kojima (2008))**.** *Let d and d' be two distinct districts. Suppose that Ch_d satisfies IRC but violates weak substitutability. Then, there exist student preferences and a path-independent admissions rule for d' such that, regardless of the other districts' admissions rules, no stable matching exists.*

Given this result, for our purposes it suffices to show the following.

Theorem 6′. *Let d be a district. There exist a set of students, their types, schools in d, and type-specific ceilings for d such that there is no district admissions rule of d that has district-level type-specific ceilings, is d-weakly acceptant, and satisfies IRC and weak substitutability.*

To show this result, consider a district d with $k_d = 2$. There are three schools c_1, c_2, c_3 in the district, each with capacity one, and four students s_1, s_2, s_3, s_4 of which two are from a different district. Students s_1 and s_2 are of type t_1 and students s_3 and s_4 are of type t_2. The district-level type-specific ceilings are as follows: $q_d^{t_1} = q_d^{t_2} = 1$.

Suppose, for contradiction, that the district admissions rule has district-level type-specific ceilings, is d-weakly acceptant, and satisfies IRC and weak substitutability.

Consider $Ch_d(\{(s_1, c_1), (s_2, c_1), (s_3, c_1), (s_4, c_1)\})$. Since types are symmetric and two students are symmetric within each type, without loss of generality, we can assume $Ch_d(\{(s_1, c_1), (s_2, c_1), (s_3, c_1), (s_4, c_1)\}) = \{(s_1, c_1)\}$ because $q_{c_1} = 1$.

Next, consider $Ch_d(\{(s_2, c_2), (s_3, c_2), (s_4, c_2)\})$. Because $q_{c_2} = 1$ and Ch_d is d-weakly acceptant, this is either equal to $\{(s_2, c_2)\}$ or $\{(s_3, c_2)\}$ (the case when it is equal to $\{(s_4, c_2)\}$ is symmetric to the case when $\{(s_3, c_2)\}$. We analyze these two cases separately.

(1) Suppose $Ch_d(\{(s_2, c_2), (s_3, c_2), (s_4, c_2)\}) = \{(s_2, c_2)\}$. Then, by IRC, we conclude that $Ch_d(\{(s_2, c_2), (s_3, c_2)\}) = \{(s_2, c_2)\}$. Next, we argue that $Ch_d(\{(s_1, c_1), (s_2, c_2), (s_3, c_2)\}) = \{(s_2, c_2)\}$. This is because the only two cases that satisfy d-weak acceptance and type-specific ceilings are $\{(s_2, c_2)\}$ and $\{(s_1, c_1), (s_3, c_2)\}$. The latter would violate weak substitutability since in that case (s_3, c_2) would be accepted in a larger set $\{(s_1, c_1), (s_2, c_2), (s_3, c_2)\}$ and rejected from a smaller set $\{(s_2, c_2), (s_3, c_2)\}$. Then, by IRC, $Ch_d(\{(s_1, c_1), (s_2, c_2), (s_3, c_2)\}) = \{(s_2, c_2)\}$ implies $Ch_d(\{(s_1, c_1), (s_2, c_2)\}) = \{(s_2, c_2)\}$. Then we note that $Ch_d(\{(s_1, c_1), (s_2, c_2), (s_3, c_1)\}) = \{(s_2, c_2), (s_3, c_1)\}$ since by weak substitutability (s_1, c_1) cannot be chosen, and therefore (s_2, c_2) and (s_3, c_1) have to be chosen due to d-weak acceptance. Next, again by weak substitutability, we note that $Ch_d(\{(s_1, c_1), (s_2, c_2), (s_3, c_1)\}) = \{(s_2, c_2), (s_3, c_1)\}$ implies $Ch_d(\{(s_1, c_1), (s_3, c_1)\}) = \{(s_3, c_1)\}$. Finally, we note that this contradicts $Ch_d(\{(s_1, c_1), (s_2, c_1), (s_3, c_1), (s_4, c_1)\}) = \{(s_1, c_1)\}$ and IRC.

(2) Suppose $Ch_d(\{(s_2, c_2), (s_3, c_2), (s_4, c_2)\}) = \{(s_3, c_2)\}$. Consider $Ch_d(\{(s_2, c_3), (s_4, c_3)\})$. Because $q_{c_3} = 1$ and Ch_d is d-weakly acceptant, this is either $\{(s_2, c_3)\}$ or $\{(s_4, c_3)\}$. We consider these two possible cases separately. These two subcases will follow similar arguments to Case (1) above and change the indices appropriately in order to get a contradiction.

    (a) Suppose $Ch_d(\{(s_2, c_3), (s_4, c_3)\}) = \{(s_2, c_3)\}$. Next, we argue that $Ch_d(\{(s_1, c_1), (s_2, c_3), (s_4, c_3)\}) = \{(s_2, c_3)\}$. This is because the only two cases that satisfy d-weak acceptance and type-specific ceilings are $\{(s_2,c_3)\}$ and $\{(s_1,c_1),(s_4,c_3)\}$. The latter would violate weak substitutability since in that case (s_4,c_3) would be accepted in a larger set $\{(s_1,c_1),(s_2,c_3),(s_4,c_3)\}$ and rejected from a smaller set $\{(s_2,c_3),(s_4,c_3)\}$. Then, by IRC, $Ch_d(\{(s_1,c_1),(s_2,c_3),(s_4,c_3)\}) = \{(s_2,c_3)\}$ implies $Ch_d(\{(s_1,c_1),(s_2,c_3)\}) = \{(s_2,c_3)\}$. Then we note that $Ch_d(\{(s_1,c_1),(s_2,c_3),(s_4,c_1)\}) = \{(s_2,c_3),(s_4,c_1)\}$ since by weak substitutability (s_1,c_1) cannot to be chosen, therefore (s_2,c_3) and (s_4,c_1) have to be chosen due to d-weak acceptance. Next, again by weak substitutability, we note that $Ch_d(\{(s_1,c_1),(s_2,c_3),(s_4,c_1)\}) = \{(s_2,c_3),(s_4,c_1)\}$ implies $Ch_d(\{(s_1,c_1),(s_4,c_1)\}) = \{(s_4,c_1)\}$. Finally, we note that this contradicts $Ch_d(\{(s_1,c_1),(s_2,c_1),(s_3,c_1),(s_4,c_1)\}) = \{(s_1,c_1)\}$ and IRC.

(b) Suppose $Ch_d(\{(s_2,c_3),(s_4,c_3)\}) = \{(s_4,c_3)\}$. Next, we argue that $Ch_d(\{(s_2,c_3),(s_3,c_2),(s_4,c_3)\}) = \{(s_4,c_3)\}$. This is because the only two cases that satisfy d-weak acceptance and type-specific ceilings are $\{(s_4,c_3)\}$ and $\{(s_2,c_3),(s_3,c_2)\}$. The latter would violate weak substitutability since in that case (s_2,c_3) would be accepted in a larger set $\{(s_2,c_3),(s_3,c_2),(s_4,c_3)\}$ and rejected from a smaller set $\{(s_2,c_3),(s_4,c_3)\}$. Then, by IRC, $Ch_d(\{(s_2,c_3),(s_3,c_2),(s_4,c_3)\}) = \{(s_4,c_3)\}$ implies $Ch_d(\{(s_3,c_2),(s_4,c_3)\}) = \{(s_4,c_3)\}$. Then we note that $Ch_d(\{(s_2,c_2),(s_3,c_2),(s_4,c_3)\}) = \{(s_2,c_2),(s_4,c_3)\}$ since by weak substitutability (s_3,c_2) cannot to be chosen, therefore (s_4,c_3) and (s_2,c_2) have to be chosen due to d-weak acceptance. Next, again by weak substitutability, we note that $Ch_d(\{(s_2,c_2),(s_3,c_2),(s_4,c_3)\}) = \{(s_2,c_2),(s_4,c_3)\}$ implies $Ch_d(\{(s_2,c_2),(s_3,c_2)\}) = \{(s_2,c_2)\}$. Finally, we note that this contradicts $Ch_d(\{(s_2,c_2),(s_3,c_2),(s_4,c_2)\}) = \{(s_3,c_2)\}$ and IRC.

□

Appendix C. Proofs of Claims

Proof of Claim 1. Since every student-school pair uniquely defines a contract, for every matching X, every school c_i, and every student s, there is at most one contract associated with s in $Ch_{c_i}(X)$. In addition, whenever a student's contract with a school c_i is chosen, her contracts with the remaining schools are included in Y_j for every $j \geq i$ by the construction of Ch_d. Hence, for every X, $Ch_d(X)$ is feasible for students. Furthermore, by assumption, $|Ch_{c_i}(X)| \leq q_{c_i}$ for each c_i. Therefore, Ch_d is feasible. □

Proof of Claim 2. Suppose that matching X is feasible for students and $x \in X_d \setminus Ch_d(X)$. There exists $i \leq n$ such that $c_i = c(x)$. Since X is feasible for students, $x \in X \setminus Y_{i-1}$ where Y_{i-1} is as defined in the construction of Ch_d. Because $x \in X_d \setminus Ch_d(X)$, $x \notin Ch_{c_i}(X \setminus Y_{i-1})$.

Then $|Ch_{c_i}(X \setminus Y_{i-1})| = q_{c_i}$ by assumption, which implies that district admissions rule Ch_d is acceptant. □

Proof of Claim 3. To show that Ch'_d is a completion of Ch_d, suppose that X is a set of contracts such that $Ch'_d(X)$ is feasible for students. By mathematical induction, we show that $Ch_{c_i}(X) = Ch_{c_i}(X \setminus Y_{i-1})$ for $i = 1, \ldots, n$, where Y_i is defined as above for $i > 1$ and $Y_0 = \emptyset$. The claim trivially holds for $i = 1$. Suppose that it also holds for $1, \ldots, i-1$. We show the claim for i. Since $Ch'_d(X)$ is feasible for students, $Ch_{c_i}(X)$ and $Ch_{c_1}(X) \cup \ldots \cup Ch_{c_{i-1}}(X)$ do not have any contracts associated with the same student. Therefore, $Ch_{c_i}(X) \cap Y_{i-1} = \emptyset$. Since Ch_{c_i} satisfies IRC, $Ch_{c_i}(X) = Ch_{c_i}(X \setminus Y_{i-1})$. As a result, $Ch_d(X) = Ch'_d(X)$, which completes the proof that Ch'_d is a completion of Ch_d.

Since all school admissions rules satisfy substitutability and LAD, so does Ch'_d. □

Proof of Claim 4. Let c be the initial school of student s and $x = (s, d, c)$. By construction, for any matching X that is feasible for students, $x \in X$ implies $x \in Ch^i_d(X)$ because c chooses x from any set of contracts and s does not have any other contract in X. Therefore, Ch^i_d respects the initial matching. □

Proof of Claim 5. Suppose that X is feasible for students. When it is the turn of school c_i, it considers X_{c_i}. Therefore, $Ch^w_d(X) = Ch_{c_1}(X_{c_1}) \cup \ldots \cup Ch_{c_k}(X_{c_k})$. Furthermore, $Ch_{c_i}(X_{c_i}) \supseteq Ch_{c_i}(\{x \in X_{c_i} | d(s(x)) = d\})$ by construction. Taking the union of all subset inclusions yields $Ch^w_d(X) \supseteq Ch^w_d(\{x \in X_d | d(s(x)) = d\})$. Therefore, Ch^w_d favors own students. □

Proof of Claim 6. To show acceptance, suppose that matching X is feasible for students and $x \in X_d \setminus Ch^b_d(X)$. There exists $i \leq n$ such that $c_i = c(x)$. Since X is feasible for students, $x \in X \setminus Y_{i-1}$ where Y_{i-1} is the set of all contracts in X associated with students who are chosen by schools $c_1, \ldots, c_{i-1}$. Because $x \in X_d \setminus Ch^b_d(X)$, x is not chosen by c_i. Then, by construction, either c_i fills its capacity or the district admits k_d students, which implies that Ch^b_d is acceptant. □

Proof of Claim 7. First, we construct a completion of Ch^b_d. Define the following district admissions rule: given a set of contracts X, when it is the turn of a school, it chooses from all the contracts in X. Each school chooses contracts using the same priority order until the school capacity is full, or the district has k_d contracts, or there are no more contracts left. Denote this admissions rule by Ch'_d. Suppose that $Ch'_d(X)$ is feasible for students. Then, by construction, $Ch'_d(X) = Ch^b_d(X)$. Therefore, Ch'_d is a completion of Ch^b_d.

Next, we show that Ch'_d satisfies LAD. Suppose that $Y \supseteq X$. Every school c_i chooses weakly more contracts from Y than X unless the number of contracts chosen from Y by the district reaches k_d. Since the number of chosen contracts from X cannot exceed k_d by construction, Ch'_d satisfies LAD.

Finally, we show that Ch'_d satisfies substitutability. Suppose that $x \in X \subseteq Y$ and $x \in Ch'_d(Y)$. Therefore, the number of contracts chosen from Y by schools preceding $c(x)$ is strictly less than k_d. Therefore, the number of contracts chosen from X by schools preceding $c(x)$ is weakly less than this number as weakly more contracts are chosen by schools preceding school $c(x)$ in Y than X. As a result, for school $c(x)$, weakly more contracts can be chosen from X than Y.

The ranking of contract x among Y in the ranking of school $c(x)$ is high enough that it is chosen from set Y. Therefore, the ranking of contract x among X in the ranking of school $c(x)$ must be high enough to be chosen from set X because weakly more contracts are chosen from X than Y for school $c(x)$. □

Proof of Claim 8. Suppose that student s is unmatched at a feasible matching X. Let t be the type of student s. Then there exists a school c such that the number of type-t students in c at X is strictly less than r_c^t because $\sum_c r_c^t = k^t$. By definition of reserves, $x = (s, c)$ satisfies $x \in Ch_{d(c)}(X \cup \{x\})$. □

Proof of Claim 9. For any set of contracts X, school c, and type t, let X_c^t denote the set of all contracts in X that are associated with school c and type-t students.

For notational simplicity, we use Ch_d instead of Ch_d^d. Consider the construction of Ch_d given in the text, but modify it by not removing contracts of students who are chosen previously. Denote this district admissions rule by Ch'_d. To show that Ch'_d is a completion of Ch_d, consider a set of contracts X and suppose that $Ch'_d(X)$ is feasible for students. Since the only difference in the constructions of Ch_d and Ch'_d is the removal of contracts of previously chosen students, it must be that $Ch'_d(X) = Ch_d(X)$. Therefore, Ch'_d is a completion of Ch_d.

To prove substitutability of Ch'_d, suppose, for contradiction, that there exist sets of contracts X and Y with $X \subseteq Y$ and a contract $x \in X$ such that $x \in Ch'_d(Y) \setminus Ch'_d(X)$. Let s and c be such that $x = (s, c)$ and $t = \tau(s)$. First, note that $|X_c^t| > r_c^t$ because $x \notin Ch'_d(X)$. Since $Y \supseteq X$, $|Y_c^t| \geq |X_c^t| > r_c^t$ is implied. Therefore, it is after all schools in d have chosen contracts based on their reserves in the algorithm describing Ch'_d that contract x is chosen by Ch'_d given Y. Let $n(X)$ and $n(Y)$ be the numbers of contracts that have been chosen by all schools before the step (call it step κ_c) at which school c chooses students beyond its reserve under X and Y, respectively. Because $x \in Ch'_d(Y)$, it follows that $n(Y) < k_d$. Therefore, for each school c', the number of contracts chosen by c' before step κ_c under Y is weakly larger than those under X, which we prove as follows:

- Suppose that school c' is processed after school c in the algorithm deciding Ch'_d. Then, by step κ_c, c' is matched with students of each type t' only up to its type-t' reserve. More formally, the numbers of type-t' students matched to c' are equal to $\min\{r^{t'}_{c'}, |X^{t'}_{c'}|\}$ and $\min\{r^{t'}_{c'}, |Y^{t'}_{c'}|\}$ under X and Y, respectively. Obviously, the latter expression is no smaller than the former expression.

- Suppose that school c' is processed before school c in the algorithm deciding Ch'_d. Recall that $n(Y) < k_d$. Therefore, for school c', it is either (i) as many as $q_{c'}$ students are matched to c' under $Ch'_d(Y)$, or (ii) for each type t', the number of type-t' students matched to c' in Y is $\min\{q^{t'}_{c'}, |Y^{t'}_{c'}|\}$. In case (i), the desired conclusion follows trivially because, given any set of contracts, the number of students matched to c' cannot exceed $q_{c'}$. For case (ii), under X, the number of type-t' students matched to c' cannot exceed $\min\{q^{t'}_{c'}, |X^{t'}_{c'}|\} \leq \min\{q^{t'}_{c'}, |Y^{t'}_{c'}|\}$. Summing up across all types, we obtain the desired conclusion.

Thus $n(X) \leq n(Y)$, so $k_d - n(X) \geq k_d - n(Y)$. Now, in step κ_c, school c will choose all the applications until either the total number of contracts chosen reaches k_d, or the total number of contracts chosen at c reaches q_c, or the number of contracts chosen at c that are associated with type t students reaches q^t_c. Given the previous fact that $k_d - n(X) \geq k_d - n(Y)$, the fact that $Y \supseteq X$, and the fact that x is chosen by c in step κ_c under Y, it has to be the case that x is also chosen by c under X in step κ_c or before. We prove this as follows:

Recall that we already established $|X^t_c| > r^t_c$. First note that at step κ_c under X and Y, for each type t, there are fewer contracts associated with school c and type-t students that remain to be processed under X than under Y ($X \subseteq Y$, and there is no contract in $X = X \cap Y$ that is processed in the reserve stage under Y but not under X), so the subset of X that should be processed in κ_c is a subset of the corresponding subset of Y. Moreover, the remaining number of contracts to be chosen before reaching the ceiling at c for each type t in step κ_c is weakly larger at X than at Y by the definition of the reserve stage. Finally, as argued above, the total number of students in the district who can still be chosen at κ_c is weakly larger under X than at Y, so whenever x is chosen under Y in this stage, x is chosen under X in this stage or the reserve stage.

This is a contradiction to the assumption that $x \notin Ch'_d(X)$.

To show that Ch'_d satisfies LAD, suppose, for contradiction, that there exist two sets of contracts X, Y with $X \subseteq Y$ and $|Ch'_d(Y)| < |Ch'_d(X)|$. Then, because $Ch'_d(Y) = \bigcup_{c:d(c)=d}(Ch'_d(Y) \cap Y_c)$ and $Ch'_d(X) = \bigcup_{c:d(c)=d}(Ch'_d(X) \cap X_c)$, there exists a school c with $d(c) = d$ such that

$$|Ch'_d(Y) \cap Y_c| < |Ch'_d(X) \cap X_c|. \tag{1}$$

Fix such c arbitrarily. Next, note that

$$|Ch'_d(Y)| < |Ch'_d(X)| \leq k_d,$$
$$|Ch'_d(Y) \cap Y_c| < |Ch'_d(X) \cap X_c| \leq q_c,$$

where the first line follows because Ch'_d is rationed by construction, and the second line also holds by construction of Ch'_d. Therefore,

$$\begin{aligned} |Ch'_d(Y) \cap Y^t_c| &= \min\{|Y^t_c|, q^t_c\} \\ &\geq \min\{|X^t_c|, q^t_c\} \\ &\geq |Ch'_d(X) \cap X^t_c|, \end{aligned} \tag{2}$$

for each type $t \in \mathcal{T}$. Because $Ch'_d(Y) \cap Y_c = \cup_{t\in\mathcal{T}}(Ch'_d(Y) \cap Y^t_c)$ and $Ch'_d(X) \cap X_c = \cup_{t\in\mathcal{T}}(Ch'_d(X) \cap X^t_c)$, Inequality (2) and the fact $Y^t_c \cap Y^{t'}_c = X^t_c \cap X^{t'}_c = \emptyset$ for any pair of types t, t' with $t \neq t'$ imply

$$|Ch'_d(Y) \cap Y_c| \geq |Ch'_d(X) \cap X_c|,$$

which contradicts Inequality (1). □

Appendix D. An Example for Theorem 4

We provide an example in which the conditions on the admissions rules stated in Theorem 4 are satisfied and, therefore, SPDA satisfies the diversity policy.

Consider a problem with two school districts, d_1 and d_2. District d_1 has school c_1 with capacity three and school c_2 with capacity two. District d_2 has school c_3 with capacity two and school c_4 with capacity one. There are seven students: students s_1, s_2, s_3, and s_4 are from district d_1, whereas students s_5, s_6, and s_7 are from district d_2. Students s_1, s_5, s_6, and s_7 have type t_1 and s_2, s_3, and s_4 have type t_2. To construct district admissions rules that satisfy the properties stated in Theorem 4, let us first specify type-specific ceilings and calculate implied floors and implied ceilings. Suppose that

$$q^{t_1}_{c_1} = 1,\ q^{t_2}_{c_1} = 2,\ q^{t_1}_{c_2} = 1,\ q^{t_2}_{c_2} = 1,$$
$$q^{t_1}_{c_3} = 2,\ q^{t_2}_{c_3} = 1,\ q^{t_1}_{c_4} = 1,\ q^{t_2}_{c_4} = 1.$$

These yield the following implied floors,[29]

$$\hat{p}^{t_1}_{d_1} = 1,\ \hat{p}^{t_2}_{d_1} = 2,\ \hat{p}^{t_1}_{d_2} = 2,\ \hat{p}^{t_2}_{d_2} = 0,$$

and implied ceilings

$$\hat{q}^{t_1}_{d_1} = 2,\ \hat{q}^{t_2}_{d_1} = 3,\ \hat{q}^{t_1}_{d_2} = 3,\ \hat{q}^{t_2}_{d_2} = 1.$$

For any type t and two districts d and d', denote $\hat{q}^t_d/k_d - \hat{p}^t_{d'}/k_{d'}$ by $\Delta^t_{d,d'}$. Using the implied floors and ceilings above, we get:

$$\begin{aligned}
\Delta^{t_1}_{d_1,d_2} &= 2/4 - 2/3 = -1/6,\\
\Delta^{t_1}_{d_2,d_1} &= 3/3 - 1/4 = 3/4,\\
\Delta^{t_2}_{d_1,d_2} &= 3/4 - 0/3 = 3/4, \text{ and}\\
\Delta^{t_2}_{d_2,d_1} &= 1/3 - 2/4 = -1/6.
\end{aligned}$$

Hence, these type-specific ceilings satisfy the condition stated in Theorem 4 that $\Delta^t_{d,d'} \leq \alpha$ for $\alpha = 0.75$.

We construct district admissions rules that have type-specific ceilings, and are rationed and weakly acceptant. Furthermore, the profile of district admissions rules accommodates unmatched students. As in Section 3.4.1, we consider type-specific reserves (as detailed below, we first fill in the reserves while applying the district admissions rule that uses type-specific reserves). Let us consider the reserves for schools as follows:

$$r^{t_2}_{c_4} = 0, \text{ and } r^t_c = 1 \text{ for all other } c, t.$$

Consider the following district admissions rule. For each district, schools and student types are ordered and each school has a linear order over students. First, schools choose contracts for their reserved seats following the master priority list until the reserves are filled or all the applicants of the relevant type are processed. Then, following the given order over schools and student types, schools choose from the remaining contracts following the linear order over students in order to fill the rest of their seats until the school

[29]To see this, note that there cannot be zero type-t_1 students in d_1 (otherwise not all type-t_1 students can be matched since there are only three spaces available for type-t_1 students in d_2). If there is one type-t_1 student in d_1, there has to be three type-t_1 students in d_2, which implies there cannot be any type-t_2 students in d_2, and this implies there will be three type-t_2 students in d_1. If there are two type-t_1 students in d_1, there have to be two type-t_1 students in d_2, which implies there is one type-t_2 student in d_2, and this implies there will be two type-t_2 students in d_1. By noting these minimum and maximum numbers, we obtain the implied reserves and implied ceilings accordingly. These bounds are achievable because it is feasible to have (i) one type-t_1 student in d_1, three type-t_1 students in d_2, zero type-t_2 students in d_2, and three type-t_2 students in d_1, and (ii) two type-t_1 students in d_1, two type-t_1 students in d_2, one type-t_2 student in d_2, and two type-t_2 students in d_1.

capacity is filled, or the district has k_d contracts, or district type-specific ceilings are filled, or there are no more remaining contracts.[30]

To give a more concrete example, suppose that the linear order over students for each school is as follows: $s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5 \succ s_6 \succ s_7$ and schools and types are ordered from the lowest index to the highest. Then, for example, we have the following:

$$Ch_{d_1}(\{(s_1, c_1), (s_2, c_1), (s_3, c_1), (s_4, c_1), (s_5, c_2), (s_6, c_2)\}) = \{(s_1, c_1), (s_2, c_1), (s_3, c_1), (s_5, c_2)\}.$$

Let us elaborate on how we determine the chosen set of contracts in the above case. School c_1 considers contracts with students s_1, s_2, s_3, and s_4. Among these students, c_1 accepts s_1 for its reserve for type t_1, and s_2 for its reserve for type t_2. Moreover, school c_2 considers contracts with students s_5 and s_6. Among these students students, c_2 accepts s_5 for its reserve for type t_2. For the remainder of seats, s_3 is accepted by c_1 since (i) c_1's type t_2 ceiling is not full, (ii) c_1's capacity is not full, and (iii) district d_1 has only three accepted contracts at this point. Next, s_4 and s_5 are rejected since d_1 has accepted four contracts at this point. This results in the chosen set of contracts presented above.

To illustrate the SPDA outcomes, consider student preferences given by the following table.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_{s_5}	P_{s_6}	P_{s_7}
c_2	c_3	c_4	c_1	c_1	c_1	c_2
$\vdots$	$\vdots$	c_2	$\vdots$	$\vdots$	c_4	c_3
		$\vdots$			$\vdots$	$\vdots$

SPDA results in the following outcome:

$$\{(s_1, c_2), (s_2, c_3), (s_3, c_2), (s_4, c_1), (s_5, c_1), (s_6, c_4), (s_7, c_3)\}.$$

District d_1 is assigned two students of both types and district d_2 is assigned two type-t_1 students and one type-t_2 student. As a result, the ratio difference for type-t_1 students between these districts is roughly 0.17, and the ratio difference for type-t_2 students is roughly 0.17. This example illustrates that the actual ratio differences can be significantly lower than the one given by Theorem 4 (0.17 versus 0.75).

[30]In Section 3.4.1, we provide a class of admissions rules that include the one we consider here. These admissions rules satisfy all of the assumptions that we make in this section.