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Abstract

We characterize Pareto optimality via *sequential* utilitarian welfare maximization: a utility vector u is Pareto optimal if and only if there exists a finite sequence of non-negative (and eventually positive) welfare weights such that u maximizes utilitarian welfare with each successive welfare weights among the previous set of maximizers. The characterization can be further related to maximization of a piecewise-linear concave social welfare function and sequential bargaining among agents à la generalized Nash bargaining. We provide conditions enabling simpler utilitarian characterizations and a version of the second welfare theorem.

Keywords: Pareto optima, weighted utilitarian welfare maximization, sequential utilitarian welfare maximization, (eventual) exposure of extreme faces, the second-welfare theorem.

JEL Numbers: C60, D60, D50.

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1 Introduction

Pareto optimality is a central concept in economics. When the utility possibility set is closed and convex, it is natural to associate each Pareto optimum with utilitarian welfare maximization under a suitably chosen welfare weights of agents. Yet, such a characterization has been so far elusive.

It is well-known that, given a closed and convex utility possibility set, which we assume throughout, every Pareto optimal utility vector maximizes some *nonnegatively* weighted sum of utilities of agents (see Mas-Colell, Whinston, and Green (1995) Proposition 16.E.2).[1] But the converse is false: not every such maximizer is Pareto optimal. To see this, suppose a society consists of two agents, 1 and 2, and the utility possibility set is given by U in Figure 1. All points on the frontier including the vertical segment maximize suitably weighted sums

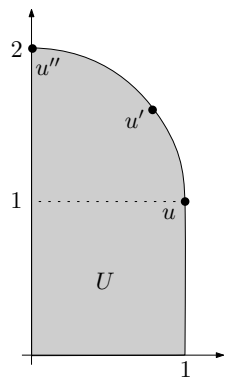


Figure 1: Utilitarian welfare maximization need not yield a Pareto optimum.

of agents' utilities within U, but not all of them are Pareto optimal. In particular, the points on the vertical segment strictly below u all maximize the utility sum with weights $\phi = (1, 0)$—i.e., only 1's utility. Yet, none of these points are Pareto optimal.

By contrast, if weights are restricted to be strictly positive for all agents, utilitarian maximization does always yield a Pareto optimum (Proposition 16.E.2 of Mas-Colell, Whinston, and Green (1995)). But again the converse is false: not every Pareto optimal outcome can be obtained in this way. In Figure 1, u' is Pareto optimal and obtained by utilitarian welfare maximization with strictly positive weights, but u and u'', which are Pareto optimal, cannot.

While positive welfare weights do not rationalize points like u in Figure 1, one may conjecture that they may in the limit; for instance, u is a limit of welfare-maximizing utility vectors with positive weights $(1, 1/n)$, as $n \to \infty$. Indeed, Arrow, Barankin, and Blackwell (1953) show that every Pareto optimal vector is a limit of a sequence of utility vectors that maximize some positively-weighted sum of utilities—a result known as the ABB theorem.[2]

[1] Propositions 16.E.2 in Mas-Colell, Whinston, and Green (1995) is stated in the context of an exchange economy. However, it is straightforward to see that the relationship holds generally as long as the utility possibility set is convex.

[2] This theorem has spawned a series of extensions and generalizations to spaces more general than Euclidean space. See Daniilidis (2000) for a survey of ABB theorems.

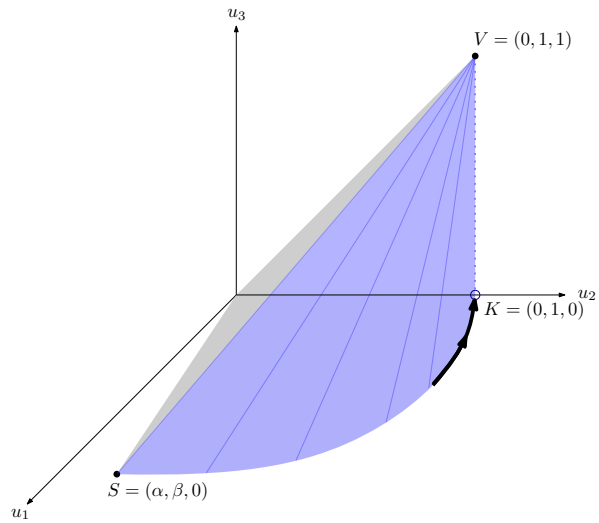


Figure 2: The "tilted cone" adapted from Arrow, Barankin, and Blackwell (1953) and Bitran and Magnanti (1979). The set is the convex hull of the portion of the unit disk centered at the origin in the u_1-u_2 plane from the point K to the point S (where $\alpha^2 + \beta^2 = 1$ with $\alpha \in (0, 1)$) and the apex point $V = (0, 1, 1)$. The blue surface, including all of its boundaries except for the dotted line, is the set of Pareto optimal utility vectors.

Unfortunately, this too does not lead to a characterization when there are more than two agents:[3] again its converse is false—namely, a limit point of such a sequence may not be Pareto optimal. To see this, suppose there are three agents, 1, 2, and 3, with possible utility profiles depicted in Figure 2. The point K is a limit of the sequence of points maximizing a positively-weighted sum of utilities (see the arrow) but is Pareto dominated, say by the point V. Figure 3 depicts Pareto optima in relationship with alternative notions of utilitarian welfare maximization.

This paper provides an exact characterization of Pareto optima in terms of utilitarian welfare maximization. In particular, our characterization views each Pareto optimal vector u as a result of multiple rounds of utilitarian welfare maximization. The characterization is easy to explain with the example in Figure 1, reproduced in Figure 4(a). In the first round, utilities are maximized within U with weights ϕ^1, which is maximized by the thick vertical segment containing u as explained before. One can interpret this as the social planner first maximizing the utility of agent 1 while disregarding the welfare of the other individual completely. Since agent 1 is indifferent among all points as long as he receives the

[3]When there are two agents, the limit $u \in U$ of any sequence $\{u^k\}$ of utilities $u^k \in U$ maximizing a positively-weighted sum of utilities is Pareto optimal, where U is the utility possibility set, assumed to be closed and convex. Let $\{\phi^k\}$ be the sequence of positive weights, normalized to be in the simplex, such that u^k maximizes $\sum_{i=1}^{2} \phi_i^k u_i^k$, and let ϕ denote its limit (say of a convergent subsequence). Clearly, u must maximize $\sum_{i=1}^{2} \phi_i u_i$. If ϕ_1 and ϕ_2 are both positive, then u is Pareto optimal, so assume without loss $\phi_1 = 1$ and $\phi_2 = 0$. Suppose for contradiction u is not Pareto optimal. Then, there must exist v such that $v_1 = u_1$ and $v_2 > u_2$. Since u^k's are all Pareto optimal, this means that $u_1^k \leq v_1 = u_1$ and $u_2^k \geq v_2 > u_2$ for all k, so u^k never converges to u, a contradiction.

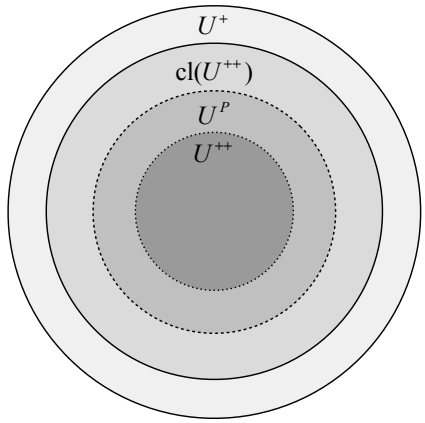
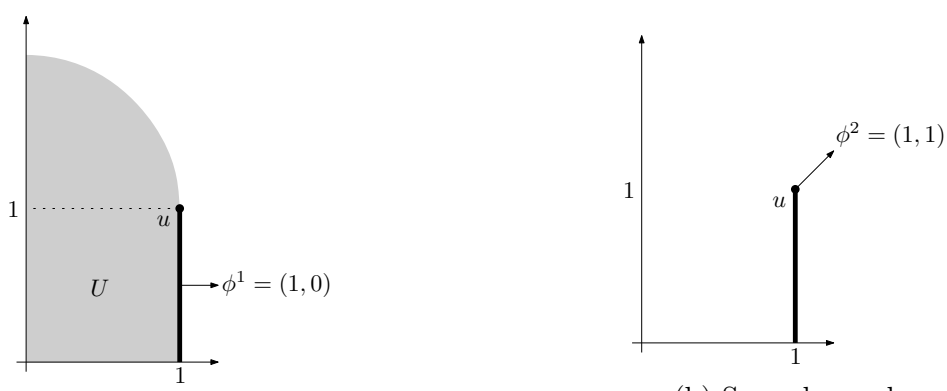


Figure 3: Alternative notions of utilitarian welfare maximization in relationship with Pareto optimality. The set U^P consists of Pareto optimal utility vectors. The set U^+ consists of utility vectors that maximizing a nonnegatively-weighted sum of utilities. The set U^{++} consists of utility vectors that maximize a positively-weighted sum of utilities. The containment $U^{++} \subset U^P \subset U^+$ is Proposition 16.E.2 in Mas-Colell, Whinston, and Green (1995). The containment $U^P \subset \mathrm{cl}(U^{++})$ is from Arrow, Barankin, and Blackwell (1953). The containment $\mathrm{cl}(U^{++}) \subset U^+$ is straightforward.



(a) First round

(b) Second round

Figure 4: Rationalizing a Pareto optimal point in two rounds of sequential utilitarian welfare maximization.

maximum utility, the social planner seeks to engage in further optimization. In the second and last round, utilities are again maximized but only within the vertical segment, now with (arbitrary) positive weights ϕ^2. The weights ϕ^2 "rationalize" u as the unique maximizer, as illustrated in Figure 4(b).[4]

More generally, our Theorem 1 asserts that a utility vector u is Pareto optimal if and only if there exists a finite sequence of nonnegative welfare weights, with the terminal weights being strictly positive for all agents, such that in each round t, u maximizes the round-t weighted sum of utilities out of those surviving from round $t-1$.

Our characterization not only fulfills a long-standing intellectual pursuit on the nature of Pareto optimality, but it can also serve some useful purposes. First, when one analyzes the behavior of Pareto optima as a set, a precise identification of the set may be necessary. For instance, one may study the comparative statics of Pareto optima—i.e., how they change as the primitives change—utilizing monotone comparative statics methods developed for optimization (e.g., Topkis (1998) and Milgrom and Shannon (1994)). These methods "track" how the set varies according to set orders such as strong or weak set order (Topkis (1998)), which requires exact identification of the set.[5] Indeed, Che, Kim, and Kojima (2019) utilize the exact characterization of the current paper to provide conditions for Pareto optima to vary monotonically with agents' preferences. Second, Pareto optimality is sometimes invoked to rationalize utilitarianism, which may otherwise be problematic due to the ordinal nature of utilities and the difficulty with interpersonal comparisons of utility (see Yaari (1981)). For this purpose, an exact characterization of Pareto optima is more appropriate than an approximate one.[6]

Our characterization builds on notions of convex geometry. Of particular interest is a special class of subsets of closed convex sets called *(extreme) faces* and the property of

[4]It is worth emphasizing that, in general, the terminal step need not identify a unique element of U but rather a set. Also, in this example, many normals work for the second-step maximization: in particular, $\phi^2 = (0, 1)$ also works. This latter choice makes this sequential maximization procedure feel reminiscent of serial dictatorship under strict preferences (Svensson, 1999), which implements every Pareto optimal allocation of agents to indivisible objects sequentially (in a one-to-one manner) by maximizing agents' welfare one at a time according to a suitably chosen serial order. Although the idea is similar in spirit, there are a couple of differences. First, a collection of individuals' joint (weighted) welfare is maximized in each round here instead of a single agent's welfare. More importantly, the sequential procedure, while practically useful, is unnecessary to find an Pareto optimal allocation in the setting of matching agents to indivisible objects. A one-round maximization of a weighted sum of utilities finds every Pareto optimum if the weights are set sufficiently differently across agents to "reflect" their serial orders.

[5]To illustrate the issue, consider two utility possibility sets U and V, corresponding to two parameters—or "before" and "after" a change of environment—and let U^P and V^P denote the sets of Pareto optima. Typically, comparing U^P and V^P directly is difficult—hence, the need for "optimization-based" characterization. Suppose for instance that V^+ dominates U^+ according to a relevant set order. This still does not imply that V^P dominates U^P according to the same set order. The same problem arises when using U^{++} or $\mathrm{cl}(U^{++})$ for the comparison.

[6]In fact, Yaari (1981) weakens Pareto optimality to *weak* Pareto optimality—i.e., no agent should be made *strictly* better off from reallocation—to achieve an exact characterization. This weaker notion of Pareto optimality, what he calls *Pareto Principle*, is "exactly" characterized by all utility vectors maximizing *nonnegatively*-weighted sums of utilities. Weak Pareto optimality, it should be noted, is not as satisfactory as Pareto optimality from a normative viewpoint.

(eventual) exposure. Importantly, it is known that any face is "eventually exposed," that is, the face coincides with the set of points that sequentially maximize *possibly negatively*-weighted sum of utilities.[7] While serving as a crucial step toward the proof, this result is not sufficient for our characterization because we require the weights to be nonnegative and eventually positive. We prove that nonnegative and eventually positive weights can be found if and only if the face consists of Pareto optimal points. The proof is nontrivial.

Our characterization has several economic interpretations. First, we provide a sense in which our characterization "reveals" preferences of the social planner. Formally, for each Pareto optimal vector u, it maximizes a piecewise-linear concave social welfare function whose linear pieces are specified by the sequence of welfare weights that rationalize u, a result reminiscent of a characterization of individual choices by Afriat (1967). Second, we establish that the sequence of welfare weights identified by our characterization result can be interpreted as the relative bargaining power of agents in a multi-round variant of generalized Nash bargaining.

With the main characterization at hand, we next ask when our characterization reduces to a simple (that is, one-round) utilitarian welfare maximization. First, if individuals' utilities are concave and monotonic in an exchange economy setting, then Pareto optimality is characterized by utilitarian welfare maximization with nonnegative weights. In this case, the two sets U^P and U^+ in Figure 3 coincide. Second, we identify a convex geometric condition under which Pareto optimality is characterized by utilitarian welfare maximization with strictly positive weights and find that the condition is met when individuals have piecewise-linear concave utility functions over a choice set that forms a convex polyhedron. In this case, the two inner sets in Figure 3 coincide. Last, we employ our methodology to generalize the second welfare theorem, showing that it holds for all Pareto optimal allocations including those in which not all types of goods are consumed by all individuals (which are excluded in the existing theorems) in exchange economies when individuals have piecewise-linear concave utility functions that satisfy a mild monotonicity property.

The remainder of the paper is organized as follows. Section 2 states the problem formally and presents our characterization and its economic interpretations. Section 3 proves the characterization result. Section 4 explores conditions that enable simple characterizations via one-round utilitarian welfare maximization. Section 5 presents a version of the second-welfare theorem. Section 6 concludes. The proofs that are not provided in the main text can be found in the appendix.

2 Statement of Main Result

Let $I = \{1, 2, \ldots, n\}$ denote a finite set of agents and $U \subset \mathbb{R}^n$ the set of possible utility profiles they may attain, or *utility possibility set*.[8] We assume that U is closed and convex. If U stems from an underlying choice space X via utility functions $(u_i)_{i \in I} : X \to \mathbb{R}^n$, then

[7] See Theorem 12.7 in Soltan (2015), reproduced as Lemma 3 below.

[8] Throughout, we use "$\subset$" to mean weak inclusion, or $\subseteq$, and likewise $\supset$ means $\supseteq$. Strict inclusion will be indicated by $\subsetneq$ and $\supsetneq$.

we let

$$U = \{(u_i)_{i\in I} \in \mathbb{R}^n \mid \exists x \in X, \forall i \in I, u_i \in [\underline{u}_i, u_i(x)]\} \tag{1}$$

where $\underline{u}_i = \inf_{x\in X} u_i(x)$.[9] That U is closed and convex is arguably a mild assumption that is satisfied if, for instance, U is induced by utility functions $(u_i)_{i\in I}$ that are upper semicontinuous and concave on a choice set X that is compact and convex.[10]

For any $u, v \in U$, we write $v \geq u$ if $v_i \geq u_i, \forall i \in I$, $v > u$ if $v \geq u$ and $v \neq u$, and $v \gg u$ if $v_i > u_i, \forall i \in I$. We say a point u in U is **Pareto optimal**, or **maximal** if there does not exist a $v \in U$ with $v > u$. Let $U^P \subset U$ denote the set of all Pareto optimal points.

For any $\phi \in \mathbb{R}^n$, consider the optimization problem:

$$\beta_\phi := \max_{u\in U} \langle \phi, u\rangle, \tag{2}$$

where $\langle \phi, u\rangle := \sum_{i=1}^n \phi_i u_i$. We call ϕ a **normal vector** since it is the normal vector of the hyperplane $\{u \in \mathbb{R}^n \mid \langle \phi, u\rangle = \beta_\phi\}$. Throughout the paper, we only consider nonzero normal vectors (i.e., $\phi \neq 0$). We say a point $u \in U$ *maximizes* the normal vector ϕ over U (or simply *maximizes* ϕ) if u is an optimal solution to (2). We call a normal vector ϕ *nonnegative* if $\phi > 0$ and *positive* if $\phi \gg 0$. For any vector $v \in \mathbb{R}^n$, the *support* of v is the set of indices where v is nonzero; i.e., $\operatorname{supp} v := \{i \in I \mid v_i \neq 0\}$. A positive ϕ has full support; i.e., $\operatorname{supp} \phi = I$.

Of particular interest are the points maximizing utilitarian welfare with nonnegative weights:

$$U^+ := \{u \in U \mid \exists \phi > 0 \text{ such that } u \in \arg\max_{u'\in U} \langle \phi, u'\rangle\}$$

and those maximizing utilitarian welfare with positive weights:

$$U^{++} := \{u \in U \mid \exists \phi \gg 0 \text{ such that } u \in \arg\max_{u'\in U} \langle \phi, u'\rangle\}.$$

As noted in Figure 3, we have $U^{++} \subset U^P \subset \operatorname{cl}(U^{++}) \subset U^+$.

Definition 1 (Sequential utilitarian welfare maximization)**.** We say $u \in U$ **sequentially maximizes** a tuple $\Phi = (\phi^1, \phi^2, \ldots, \phi^T)$ of normals over U if

$$u \in U^t := \arg\max_{u'\in U^{t-1}} \langle \phi^t, u'\rangle, \text{ for each } t = 1, \ldots, T, \tag{3}$$

where $U^0 = U$. We say $u \in U$ **sequentially maximizes utilitarian welfare** over U if there is a tuple $\Phi = (\phi^1, \phi^2, \ldots, \phi^T)$ of $T \leq n$ nonnegative normals (i.e., $\phi^t > 0$) with $\phi^T \gg 0$ such that u sequentially maximizes Φ.

[9] To be precise, the utility possibility set is often defined as $\{u \in \mathbb{R}^n \mid u = (u_i(x))_{i\in I} \text{ for some } x \in X\}$, which differs from (1). However, the two sets share the same set of maximal points since those points are on the frontier. Thus, formulating the set U either way makes no difference for our results while the current formation facilitates our analysis.

[10] Note that compactness and convexity of the choice set X are satisfied if, for instance, all lotteries of social outcomes are feasible.

Theorem 1. Let U be a closed convex set. Then u is Pareto optimal if and only if u sequentially maximizes utilitarian welfare over U.

This theorem states that even though simple utilitarian welfare maximization may not characterize Pareto optima, sequential utilitarian welfare maximization does. As discussed in the introduction, this characterization is useful in its own right—as exemplified by Che, Kim, and Kojima (2019)—but it also allows for interesting interpretations of Pareto optima as follows.

First, even though the maximization of a utilitarian welfare function does not characterize Pareto optimal points, our sequential characterization identifies a nonlinear welfare function that a Pareto optimal point maximizes. In fact, the tuple of normals Φ identified by Theorem 1 parameterizes a piecewise-linear concave (PLC) social welfare function the Pareto optimal point maximizes.[11]

Corollary 1. Let $u \in U$ be a Pareto optimal point. Then

$$u \in \arg\max_{u' \in U} W(u'),$$

where $W(u') := \min_{t \in \{1,\dots,T\}} \langle \phi^t, (u' - u) \rangle$ with $\phi^t > 0, \forall t$, and $\phi^T \gg 0$, where Φ is a tuple of normals identified in Theorem 1. Moreover, any point in $\arg\max_{u' \in U} W(u')$ is Pareto optimal.

Proof. By Theorem 1, there is a tuple $(\phi^1, \dots, \phi^T)$ with which u sequentially maximizes utilitarian welfare over U. To show the first part, let $u' \in U$ and $U^1, \dots, U^T$ be the sets defined in Definition 1. If $u' \in U^T$, then $\langle \phi^t, u' \rangle = \langle \phi^t, u \rangle$ for every t, so $W(u') = 0 = W(u)$. If $u' \notin U^T$, then there exists $t \in \{1, \dots, T\}$ such that $\langle \phi^t, u' \rangle < \langle \phi^t, u \rangle$, so $W(u') = \min_{t \in \{1,\dots,T\}} \langle \phi^t, (u' - u) \rangle < 0 = W(u)$. Therefore $u \in \arg\max_{u' \in U} W(u')$.

To show the second part, let $v \in \arg\max_{u' \in U} W(u')$. From the proof of the first part, $v \in U^T$. Therefore, by Theorem 1, v is Pareto optimal. □

Second, one can think of each Pareto optimum as emerging from a sequence of negotiations among individuals, where ϕ^t shapes the agents' relative bargaining power in round t negotiation. This idea can be fleshed out in the following way. First, we assume that each agent has a disagreement utility, normalized as zero, that is less than any Pareto optimal utility: $u \in U^P$ means $u \gg 0$. For a partition $\mathcal{I} = \{I^1, \dots, I^T\}$ of I, imagine that the agents engage in a sequence of bargaining: in round 1, agents in I^1 bargain from U to a set $V^1 \subset U$, and in round $t = 2, \dots, T$, agents in set I^t bargain from V^{t-1} to a set V^t. The bargaining protocol in each round t is a generalized Nash bargaining game (Kalai, 1977) in which each agent $i \in I^t$ has a bargaining power $\psi_i > 0$ such that $\sum_{i \in I^t} \psi_i = 1$ and a disagreement payoff

[11] This interpretation is reminiscent of Afriat's theorem that also constructs a PLC function to rationalize observed consumer choices. In Afriat's theorem, however, every normal vector used in the construction is strictly positive since it coincides with the price vector associated with each choice. Also, in Afriat's theorem, each choice associated with each normal vector is rationalized by the PLC utility function whereas, in our result, it is only the choices in the last step (i.e., U^T) that are rationalized by the constructed PLC function.

0. More specifically, for each $t \geq 1$ we set $V^t := \arg\max_{u \in V^{t-1}} \prod_{i \in I^t} u_i^{\psi_i}$ and let the solution v of the bargaining be defined by $v_i = v_i^t$ for each $i \in I^t$ where v^t is an arbitrary element of V^t (it turns out that $v_i^t = w_i^t$ for every $i \in I^t$ if $v^t, w^t \in V^t$). We call such a bargaining protocol a *sequential generalized Nash bargaining game*.

Corollary 2. Let $u \in U$ be a Pareto optimal point and $\Phi = (\phi^1, \ldots, \phi^T)$ be the normals with which u sequentially maximizes utilitarian welfare over U. Then, u is a solution to a sequential generalized Nash bargaining game, where the round-t bargainers are $I^t = \operatorname{supp} \phi^t \setminus (\bigcup_{t' \leq t-1} I^{t'})$ (with $I^0 = \emptyset$) and their bargaining powers are $\psi_i = \frac{\phi_i^t u_i}{\sum_{j \in I^t} \phi_j^t u_j}$ for each $i \in I^t$.

Proof. Since the sets of bargainers are disjoint across rounds, the game is separated into a collection of generalized Nash bargaining games. We argue inductively that $V^t = \{u' \in U \mid u'_i = u_i, \forall i \in \cup_{\ell=0}^{t-1} I^\ell\} \subset U^{t-1}$, where U^{t-1} is defined in (3).

To prove the claim, consider round $t = 1$. We wish to prove that $V^1 = \{u' \in U \mid u'_i = u_i, \forall i \in I^1\}$ forms a solution to the round-1 generalized Nash bargaining game, or more specifically, it solves:

$$[NB_1] \qquad \max_{u' \in U} W^1(\{u'_i\}_{i \in I^1}) := \prod_{i \in I^1} (u'_i)^{\psi_i}.$$

If $|I^1| = 1$, then u_i, $\{i\} = I^1$, must maximize u'_i within U, and V^1 is clearly the set of optimal solutions for $[NB_1]$. We thus assume $|I^1| > 1$. To solve $[NB_1]$ for this case, consider a relaxed program:

$$[NB'_1] \qquad \max_{u' \in \mathbb{R}^n} W^1(\{u'_i\}_{i \in I^1}) \text{ s.t. } \langle \phi^1, u' \rangle \leq \langle \phi^1, u \rangle.$$

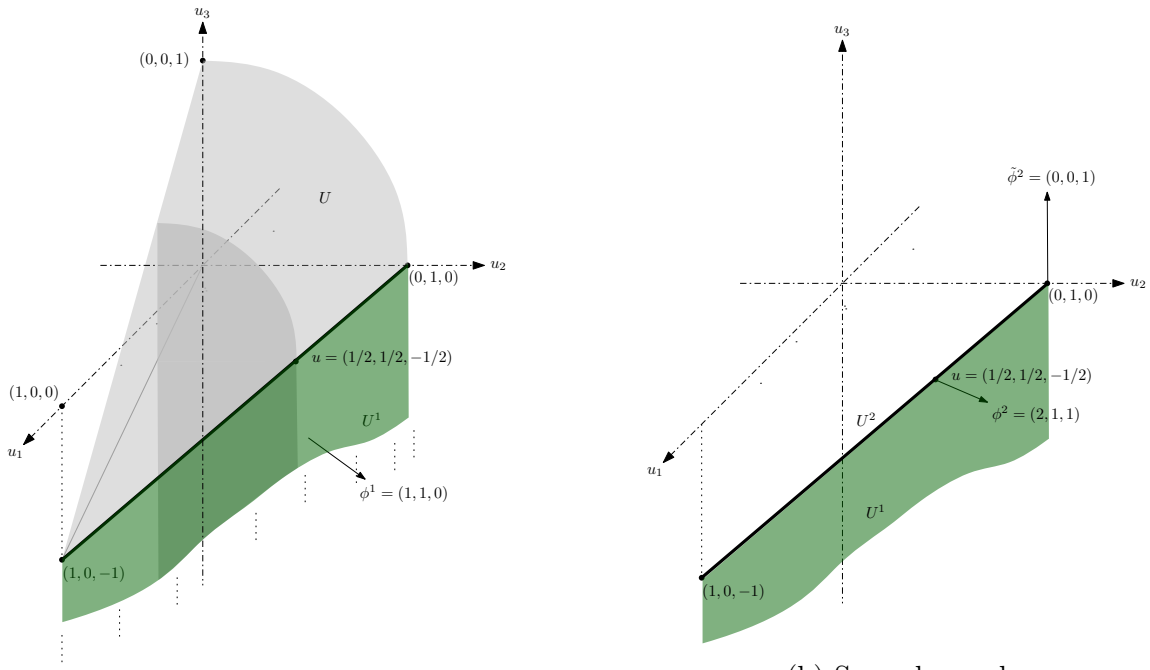
If V^1 is the set of optimal solutions for $[NB'_1]$, then V^1 must also be the set of optimal solutions for $[NB_1]$ since $V^1 \subset U \subset \{u' \in \mathbb{R}^n \mid \langle \phi^1, u' \rangle \leq \langle \phi^1, u \rangle\}$. Since $[NB'_1]$ has a strictly concave objective function and a linear constraint with respect to $(\{u'_i\}_{i \in I^1})$, first-order conditions completely characterize its optimal solution. They yield, for any $i, j \in I^1$,

$$\frac{\phi_i^1}{\phi_j^1} = \frac{\psi_i u'_j}{\psi_j u'_i} \Rightarrow \frac{u'_i}{u'_j} = \frac{u_i}{u_j}.$$

Suppose $u'_j \neq u_j$ for some $j \in I^1$. Then, either $u'_i > u_i$ for all $i \in I^1$ or $u'_i < u_i$ for all $i \in I^1$. In the former case, the solution violates $\langle \phi^1, u' \rangle \leq \langle \phi^1, u \rangle$. In the latter case, $W^1(\{u'_i\}_{i \in I^1}) < W^1(\{u_i\}_{i \in I^1})$ since W^1 is strictly increasing. It follows that the optimal solutions have $u'_i = u_i$ for all $i \in I^1$—i.e., V^1 forms the optimal solutions for $[NB_1]$.

Since the first round bargaining pins down the utilities for all $i \in I^1$, the second round bargaining by I^2 occurs over a feasible set $V^1 = \{u' \in U \mid u'_i = u_i, \forall i \in I^1\}$. The argument from this point onwards is analogous, noting that $V^t \subset V^{t-1} \subset U^{t-1}$. □

Remark 1. As mentioned earlier (in Footnote 4), there can be different sets of normals that characterize the same maximal point: in Figure 4 for instance, $\phi^1 = (1, 0)$ and $\phi^2 = (0, 1)$ also



(a) First round

(b) Second round

Figure 5: Theorem 1 cannot be modified so that the supports of the normals in Φ partition I.

work, selecting the vertical edge containing u from U and then u from this edge, respectively. This characterization has a special appeal since it lends a *serial-dictatorship* interpretation to Pareto optimal choice: in the example of Figure 4, agents 1 and 2 play as a dictator in the first and second step, respectively. One way to generalize such a characterization beyond the two-agent case would be to construct a finite set of normals $\Phi = (\phi^1, \ldots, \phi^T)$ where the supports of ϕ^t for $t = 1, \ldots, T$ partition I, so agents in $\operatorname{supp} \phi^t$ play jointly as step-t dictators.

However, such a partition characterization of Pareto optimality does not hold when there are more than two agents. Consider the maximal point $u = (1/2, 1/2, -1/2)$ of the closed convex set U in Figure 5. In Step 1, the point u must maximize nonnegative normals over U of the form $\phi^1 = (\alpha, \alpha, 0)$ where $\alpha > 0$ (in the figure we take $\alpha = 1$), which entails U^1 (shaded face weakly below the thick line segment) as the set of maximizers. For the normals to partition I, the Step 2 normal should be of the form $(0, 0, \beta)$ for some $\beta > 0$ (we take $\beta = 1$ in the normal $\tilde{\phi}^2$ in the figure). However, u does not maximize such a normal out of U^1 in Step 2. Any normal maximized by u in Step 2 must assign positive weights to at least one of the first two components, violating the partition structure. However, a strictly positive normal $\phi^2 = (2, 1, 1)$ is maximized by u among the points in U^1 (indeed, every point in the thick line segment U^2 maximizes ϕ^2).

3 Proof of Theorem 1

We now prove Theorem 1. The "if" direction is rather straightforward:

Proof of the "if" part of Theorem 1. Let u sequentially maximize utilitarian welfare over U. Then, there is a tuple $(\phi^t)_{t=1}^T$ that is sequentially maximized by u and satisfies $\phi^t > 0, \forall t$ and $\phi^T \gg 0$. Suppose to the contrary that u is not maximal so there is a point $v \in U$ such that $v > u$. Observe then that $\langle \phi^t, v \rangle \geq \langle \phi^t, u \rangle, \forall t$, which implies $v \in U^t, \forall t$, since $u \in U^t, \forall t$. In particular, $u, v \in U^{T-1}$. However, $\phi^T \gg 0$ and $v > u$ imply $\langle \phi^T, v \rangle > \langle \phi^T, u \rangle$, which contradicts that $u \in U^T$. □

The "only if" direction of the proof of Theorem 1 is nontrivial. We begin with some preliminaries necessary for the proof. Only statements of results in this section are given. Proofs are either found in standard references (such as Soltan (2015)) or placed in the appendix when not shown elsewhere.

3.1 Preliminaries

Let us first introduce a few concepts that are crucial for our analysis. A **face** of U is a nonempty convex subset F of U with the property that if $u \in F$ and $u = \alpha v + (1-\alpha)w$ for some $0 < \alpha < 1$ and $v, w \in U$ then it must be that $v, w \in F$. That is, F is a face of a convex set if none of its elements are convex combinations of elements that lie outside of F. A **proper face** of U is a face of U that is a proper subset of U. A face F is an **exposed face** of U if there is a normal $\phi \in \mathbb{R}^n$ such that $F = \arg\max_{u \in U} \langle \phi, u \rangle$. In this case, we say that ϕ exposes F out of U. A face need not be exposed, as can be seen in Figure 1, where u forms a singleton face (and is thus also an extreme point) that is not exposed. The face U^1 in Figure 5 is an example of a higher-dimensional non-exposed face.

For any convex subset G of U, its *relative interior* $\mathrm{ri}(G)$ is the set of all $u \in G$ such that for every $u' \in G$ there exists $\lambda > 0$ such that $u + \lambda(u - u') \in G$.

The following lemma shows a face structure of a convex set that is interesting in itself and useful for our analysis.

Lemma 1 (Corollary 11.11(a) in Soltan (2015))**.** For a convex set $U \subseteq \mathbb{R}^n$, the collection of relative interiors of faces—that is, $\{\mathrm{ri}(F) : F \text{ is a face of } U\}$—forms a partition of U.

The next lemma shows that maximal points "come in faces."

Lemma 2. Let u be a maximal point of a closed convex set U in the relative interior of a face F of U. Then, every point in F is maximal.

Accordingly, we say a face is *maximal* if all of its elements are maximal. Importantly for our purpose, Lemma 1 and Lemma 2 imply that every maximal point of U belongs to a relative interior of a unique maximal face of U (possibly U itself).

The next result provides a key step of our argument: every face, possibly non-exposed, is *eventually* exposed.[12]

Lemma 3 (Theorem 12.7 in Soltan (2015))**.** Let $U \subset \mathbb{R}^n$ be a convex set and F be a nonempty proper face of U. There is a sequence of convex sets $(G^t)_{t=0}^T$ such that

$$F = G^T \subset G^{T-1} \subset \cdots \subset G^1 \subset G^0 = U, \tag{4}$$

where G^t is a nonempty proper exposed face of G^{t-1} for each $t = 1, \ldots, T$.

This lemma, which will be a crucial element of our proof, is already illustrated in the Introduction. In Figure 4, the singleton face u is exposed in two steps: the vertical segment is exposed first by a normal $(1, 0)$, and then u is exposed by normal $(1, 1)$ (among many others) out of that vertical segment. This lemma is not enough for our result, however, as it is silent about any additional properties on the normals that expose the sequence of faces. Crucially, our characterization requires the normals to be nonnegative and eventually positive.

For these additional features, we need to introduce a set of analytical tools. Let J be any subset of the index set I and let χ^J denote the vector whose i-th coordinate is equal to 1 for every $i \in J$ and equal to 0 for every $i \notin J$. When J is the singleton $\{i\}$ we simplify $\chi^{\{i\}}$ to χ^i. A convex set U is **downward closed in coordinates** $J \subset I$ if, for all $u \in U$ and all $\tau \geq 0$, $u - \tau\chi^K \in U$ for any subset K of J. A convex set that is downward closed in coordinates I is simply called **downward closed**. The **downward closure** of a closed convex set U is the downward closed set $\mathrm{dc}(U) := \bigcup_{u \in U}(u - \mathbb{R}^n_+)$. It is straightforward to see that $\mathrm{dc}(U)$ is closed and convex if U is closed and convex.

One useful feature of downward closure is that it preserves maximal elements and thus maximal faces.

Lemma 4. The set of maximal elements of a closed convex set coincides with that of its downward closure. If F is a maximal face of U then F is a maximal face of $\mathrm{dc}(U)$.

Crucially for our arguments, supporting hyperplanes of downward-closed sets must have nonnegative normals.

Lemma 5. For any closed convex set U that is downward closed in coordinates $J \subset I$, any supporting hyperplane of U has a normal ϕ with $\phi_j \geq 0, \forall j \in J$.

The next lemma shows that downward closedness is preserved under maximization for the coordinates to which the normal assigns zero weights.

Lemma 6. Let F be a face of a closed convex set U that is downward closed in coordinates $J \subset I$. If ϕ exposes F out of U, then F is downward closed in coordinates $J \setminus \operatorname{supp} \phi$.

Armed with these preliminary observations, we are now ready to prove the "only if" direction of Theorem 1.

[12] Theorem 5 of Lopomo, Rigotti, and Shannon (2019) proves the same result for singleton faces F; i.e., extreme points.

3.2 Proof of the "only if" direction

Fix any maximal point u of U. We wish to show that u sequentially maximizes utilitarian welfare over U. The proof consists of several steps.

Step 1. There exists a unique face F of $\mathrm{dc}(U)$ such that $u \in \mathrm{ri}(F)$. All points of F are maximal in $\mathrm{dc}(U)$.

Proof. By Lemma 4, u is a maximal point of $\mathrm{dc}(U)$. By Lemma 1 there is a unique face F of $\mathrm{dc}(U)$ which contains u in $\mathrm{ri}(F)$. By Lemma 2, every point of F is maximal in $dc(U)$, as desired. □

Step 2. The face F (containing u) is a proper face of $\mathrm{dc}(U)$.

Proof. If not, we must have $F = \mathrm{dc}(U)$. Pick any $u' \in \mathrm{dc}(U)$. Then $u'' = u' - \epsilon(1, 1, \ldots, 1)$ is also in $\mathrm{dc}(U)$ by the downward closure property. Clearly, u'' is not a maximal point of $\mathrm{dc}(U)$ and cannot belong to F by Step 1, a contradiction. □

Step 3. There exists a sequence of convex sets $(G^t)_{t=0}^T$ of $\mathrm{dc}(U)$ such that G^t is a proper exposed face of G^{t-1} for $t = 1, \ldots, T$, where $G^0 = \mathrm{dc}(U)$, $G^T = F$, and $T \leq n$.

Proof. Since F is a proper face of $\mathrm{dc}(U)$ by Step 2, the result follows from Lemma 3. For any set V, let $\dim(V)$ denote its dimension.[13] If V' is a proper face of convex set V, then $\dim(V') < \dim(V)$ by Theorem 11.4 in Soltan (2015). Thus, we have $T \leq n$ since $\dim(G^{t-1}) < \dim(G^t)$ and since $\dim(G^0) = \dim(\mathrm{dc}(U)) = n$. □

Step 4. There exists a tuple $\Phi = (\phi^1, \ldots, \phi^T)$ such that for each $t = 1, \ldots, T$,

$$G^t = \arg\max_{x \in G^{t-1}} \langle \phi^t, x \rangle,$$

where $\phi^t > 0$, $\phi^T \gg 0$, and $\operatorname{supp} \phi^t \supset \operatorname{supp} \phi^{t-1}$.

Proof. By Step 3, there exists a sequence of normals $\Psi = (\psi^1, \ldots, \psi^T)$ such that, for each $t = 1, \ldots, T$, ψ^t exposes G^t out of G^{t-1}. We construct $\Phi = (\phi^1, \ldots, \phi^T)$ with the stated properties.

The construction is recursive. First, since $G^0 = \mathrm{dc}(U)$, by Lemma 5, $\phi^1 := \psi^1$ is non-negative. For an inductive hypothesis, suppose that there are ϕ^k, $k = 1, .., t-1$, with the stated properties and that for each $k = 1, \ldots, t-1$, G^k is downward-closed in coordinates $J^k := \{i \in I \mid \phi_i^k = 0\} = I \setminus \operatorname{supp} \phi^k$. Note that $J^{t-1} \subset J^{t-2} \subset \cdots \subset J^0 := I$. From now, we construct ϕ^t and show that G^t is downward-closed in coordinates $J^t = \{i \in I \mid \phi_i^t = 0\}$.

First, since ψ^t is a normal for the supporting hyperplane of G^{t-1} and G^{t-1} is downward-closed in coordinates J^{t-1}, Lemma 5 implies that $\psi_j^t \geq 0$ on coordinates $j \in J^{t-1}$. Consider

[13] The dimension $\dim(V)$ of a convex subset V of U, including one of U's faces, is defined by the dimension of its affine hull: $\mathrm{aff}(V) := \{\sum_{j=1}^k \alpha_j v^j \mid k \in \mathbb{N}, v^j \in V, \alpha_j \in \mathbb{R}, \sum_{j=1}^k \alpha_j = 1\}$.

next $i \in \operatorname{supp} \phi^{t-1} = I \setminus J^{t-1}$. For such i, it is indeed possible for ψ_i^t to be negative. But noting $\phi_i^{t-1} > 0$ for such i, we define

$$\phi^t = \lambda^t \phi^{t-1} + \psi^t,$$

where $\lambda^t > \max_{i \in \operatorname{supp} \phi^{t-1}} |\psi_i^t| / \phi_i^{t-1}$ is a (sufficiently large) positive scalar. Given this construction, $\phi_i^t \geq 0$ for all $i \in I$ and $\phi_i^t > 0$ for all $i \in \operatorname{supp} \phi^{t-1}$, that is, $\operatorname{supp} \phi^t \supset \operatorname{supp} \phi^{t-1}$.

Let us show that ϕ^t exposes G^t out of G^{t-1}. To this end, let $M^t := \max_{x \in G^{t-2}} \langle \phi^{t-1}, x \rangle$. For all $x \in G^{t-1}$, we have

$$\langle \phi^t, x \rangle = \lambda^t \langle \phi^{t-1}, x \rangle + \langle \psi^t, x \rangle = \lambda M^t + \langle \psi^t, x \rangle,$$

since $\langle \phi^{t-1}, x \rangle = M^t$ for all $x \in G^{t-1}$. Henceforth,

$$\arg \max_{x \in G^{t-1}} \langle \phi^t, x \rangle = \arg \max_{x \in G^{t-1}} \langle \psi^t, x \rangle = G^t.$$

Since G^{t-1} is downward-closed in coordinates J^{t-1} and ϕ^t exposes G^t out of G^{t-1}, Lemma 6 implies that G^t is downward-closed in coordinates $J^{t-1} \setminus \operatorname{supp} \phi^t = (I \setminus \operatorname{supp} \phi^{t-1}) \setminus \operatorname{supp} \phi^t = I \setminus \operatorname{supp} \phi^t = J^t$, where the penultimate equality holds since $\operatorname{supp} \phi^{t-1} \subset \operatorname{supp} \phi^t$.

It remains to show that ϕ^T is positive. Supposing not, there must be some $i \in I$ such that $\phi_i^t = 0$ for all $t = 1, \ldots, T$, so $i \in J^t$ for all $t = 1, \ldots, T$. Then, Lemma 6 implies that for all $t = 1, \ldots, T$, G^t is downward-closed in coordinate i, which contradicts the fact that $G^T = F$ is maximal. □

We have so far shown that u sequentially maximizes welfare over $\mathrm{dc}(U)$. We now prove the main result: u sequentially maximizes welfare over U. To this end, the following last step suffices.

Step 5. u sequentially maximizes utilitarian welfare over U.

Proof. Recall a sequence of normals Φ from Step 4. Let $U^0, U^1, \ldots, U^T$ be convex subsets of U such that, for each $t = 1, \ldots, T$, U^t is the face of U^{t-1} exposed by normal ϕ^t, i.e.,

$$U^t = \arg \max_{x \in U^{t-1}} \langle \phi^t, x \rangle,$$

where $U^0 := U$. It suffices to prove that $U^T = F$, as this will prove that u sequentially maximizes utilitarian welfare over U.

To this end, it suffices to prove $F \subset U^t \subset G^t$ for each $t = 0, 1, 2, \ldots, T$. We proceed inductively for the proof. First, note that the claim is trivially true for $t = 0$ because $U^0 := U \subset dc(U) := G^0$ and $F \subset U = U^0$ by definition. Now, suppose that the claim holds for t. We show (i) $F \subset U^{t+1}$ and (ii) $U^{t+1} \subset G^{t+1}$ as follows.

For (i), fix any point v in F. Then, since $F \subset G^{t+1}$ and ϕ^{t+1} exposes G^{t+1} out of G^t, we have $\langle \phi^{t+1}, v \rangle \geq \langle \phi^{t+1}, w \rangle$ for every $w \in G^t$. Because $U^t \subset G^t$ by the inductive assumption,

$$\langle \phi^{t+1}, v \rangle \geq \langle \phi^{t+1}, w \rangle \tag{5}$$

for every $w \in U^t$. Moreover, $v \in U^t$ by the assumption that $F \subset U^t$. This fact, combined with (5), implies that ϕ^{t+1} is maximized by v over U^t and so $v \in U^{t+1}$, since ϕ^{t+1} exposes U^{t+1} out of U^t. This holds for every $v \in F$ and so $F \subset U^{t+1}$, implying (i) holds for $t+1$.

As for (ii), fix any point v in U^{t+1}. By (i), we know that

$$\langle \phi^{t+1}, v \rangle = \langle \phi^{t+1}, w \rangle \tag{6}$$

for any $w \in F$, since U^{t+1} is exposed by ϕ^{t+1} and F is a subset of U^{t+1}. Also, by definition of G^{t+1} and the fact $F \subset G^{t+1}$ by construction, we know that

$$\langle \phi^{t+1}, w \rangle \geq \langle \phi^{t+1}, z \rangle \tag{7}$$

for any $w \in F$ and $z \in G^t$. Combining (6) and (7) implies that $\langle \phi^{t+1}, v \rangle \geq \langle \phi^{t+1}, z \rangle$ for any $z \in G^t$. This, and the fact that $v \in G^t$ (which immediately follows from $v \in U^{t+1} \subset U^t \subset G^t$), means that $v \in G^{t+1}$. Since this holds for any $v \in U^{t+1}$, we can conclude that $U^{t+1} \subset G^{t+1}$, so (ii) holds for $t+1$.

This completes the induction and establishes the result. □

4 Pareto Optimality and Simple Utilitarianism

The previous section provided a precise rationalization of Pareto optimality in terms of sequential utilitarian welfare maximization. The question remains, however, as to when Pareto optimality coincides with the simpler notions of utilitarianism: nonnegative and positive. In particular, this section explores when U^P coincides with either U^+ or U^{++}. These conditions follow naturally from our characterization in Theorem 1.

4.1 Pareto Optimality (U^P) and Nonnegative Utilitarianism (U^+)

One case where $U^P = U^+$ is rather well-understood in the literature; the case with *strict* convexity. Formally, we say that a set $U \subset \mathbb{R}^n$ is **strictly convex** if $u, v \in U$ and $\lambda \in (0,1)$ imply $\lambda u + (1-\lambda)v \in \text{int}(U)$, where int denotes the interior of a set. It is well-known that if U is closed and strictly convex, then $U^P = U^+$.[14]

Another setting of interest is the following exchange economy environment. Let there be m types of goods with some integer $m > 0$. For each $k \in \{1, \ldots, m\}$, let $\bar{e}^k > 0$ be the total supply of type-k goods in the environment. Let $\bar{e}$ denote the vector $(\bar{e}^k)_{k=1}^m$. Each alternative $x = (x_i)_{i \in I}$, $x_i = (x_i^k)_{k=1}^m \in \mathbb{R}_+^m$, specifies consumption bundle x_i for each $i \in I$. A profile of consumption bundles x is said to be feasible if and only if $\sum_{i \in I} x_i \leq \bar{e}$. In this context,

[14] This result appears to be a folk result, and we do not know who first made this observation. For completeness, we provide a proof sketch here. First, the fact $U^P \subset U^+$ follows from Theorem 1. To prove the set inclusion relationship in the opposite direction, suppose for contradiction that $u \in U$ maximizes a nonnegative normal ϕ but there exists $v \in U$ such that $v > u$. Then, for any $\lambda \in (0,1)$, a point $w := \lambda u + (1-\lambda)v$ satisfies $\langle \phi, w \rangle = \lambda \langle \phi, u \rangle + (1-\lambda)\langle \phi, v \rangle \geq \langle \phi, u \rangle$. Because U is strictly convex, $w \in int(U)$ and hence there exists $x \in U$ such that $x_i > w_i$ for every $i \in I$. Therefore we obtain $\langle \phi, x \rangle > \langle \phi, w \rangle \geq \langle \phi, u \rangle$, a contradiction.

the choice set X is defined as the set of all feasible profiles of consumption bundles. Each individual $i \in I$ is endowed with a utility function $u_i : \mathbb{R}^m_+ \to \mathbb{R}$.

Suppose that, for each $i \in I$, the utility function $u_i : \mathbb{R}^m_+ \to \mathbb{R}$ is concave. We say $u_i(\cdot)$ is **strictly monotonic** if $u_i(x_i) > u_i(y_i)$ for every $x_i, y_i \in \mathbb{R}^m_+$ with $x_i > y_i$. Without loss of generality, we normalize $u_i(0) = 0$ for all $i \in I$: note that $u_i(0) = \min_{x_i \in \mathbb{R}^m_+} u_i(x_i)$ if $u_i(\cdot)$ is strictly monotonic because $0 \in \mathbb{R}^m_+$ is the smallest element of $\mathbb{R}^m_+$. The utility possibility set U is then defined as in (1) with $\underline{u}_i = 0, \forall i \in I$. Let us refer to a tuple $\mathcal{E} := (I, \bar{e}, (u_i(\cdot))_{i \in I})$ as an economy.

Theorem 2. Let $\mathcal{E}$ be an economy where $u_i(\cdot)$ is strictly monotonic for each $i \in I$. The set of Pareto optimal points coincides with those that maximize nonnegative normals, i.e., $U^P = U^+$.

We note a subtle but crucial difference between this result and existing results in general equilibrium theory. In the latter, it is customary to restrict attention to a subset of Pareto optimal points that are supported by an alternative with the additional restriction that every individual receives a strictly positive amount of every type of good, i.e., $x_i^k > 0$ for each $i \in I$ and $k \in \{1, \ldots, m\}$ (see Argenziano and Gilboa (2015) for instance). Theorem 2, by contrast, does not make any such restriction and characterizes the entire set of Pareto optimal points.

It is worth noting that strict monotonicity of utility function differs from local nonsatiation, a commonly assumed condition in general equilibrium theory (see, for instance, Section 16.C in Mas-Colell, Whinston, and Green (1995)). We say that a utility function $u_i : \mathbb{R}^m_+ \to \mathbb{R}$ is *locally nonsatiated* if, for any $x_i \in \mathbb{R}^m_+$ and $\epsilon > 0$, there exists $y_i \in \{y_i \in \mathbb{R}^m_+ \mid |y_i - x_i| < \epsilon\}$ with $u_i(y_i) > u_i(x_i)$. The following example shows that the characterization in Theorem 2 does not hold if we weaken the strict monotonicity to local nonsatiation.

Example 1. Suppose that there are two individuals, 1 and 2, as well as two types of divisible goods 1 and 2 with unit supply each, i.e., $\bar{e} = (1, 1)$. Utility functions of the individuals are given by

$$u_1(x^1, x^2) = x^1,$$
$$u_2(x^1, x^2) = \sqrt{x^1} + x^2.$$

Note that these utility functions satisfy local nonsatiation, but $u_1(\cdot)$ fails strict monotonicity as it is constant in x^2. The utility possibility set coincides with U in Figure 1, so U^P does not coincide with U^+.

4.2 Pareto Optimality (U^P) and Positive Utilitarianism (U^{++})

The goal of this subsection is to discover natural conditions for U^P to coincide with U^{++}. The following corollary, which follows easily from the proof of Theorem 1, is the key to our investigation.

Corollary 3. If u is a maximal element of U that lies in the relative interior of an exposed face of $\mathrm{dc}(U)$ then u maximizes a positive normal over U.

Proof. In the proof of the "only if" part of Theorem 1 in Section 3.2, if u is a maximal element of U that lies in the relative interior of an exposed face of $\mathrm{dc}(U)$, then $T = 1$ in Step 3 and by Step 4 we know ϕ^1 is positive. Hence, $\Phi = (\phi^1)$ and so by Step 5, we conclude that u maximizes the positive normal ϕ^1 over U. □

This corollary allows us to prove the following characterization of when $U^P = U^{++}$. The proof uses the concept of a normal cone and some of its properties, details of which are found in Appendix C.

Theorem 3. Let U be a closed convex set. Then $U^P = U^{++}$ if and only if every maximal element of U belongs to some exposed maximal face of $\mathrm{dc}(U)$.

We now discuss a few of the nuances in the statement of Theorem 3. First, the condition cannot be weakened so that every maximal element of U simply lies in a (potentially non-maximal) exposed face of $\mathrm{dc}(U)$. Consider our canonical example in Figure 1. The point u lies on an exposed face of $\mathrm{dc}(U)$ but this face is not a maximal face of $\mathrm{dc}(U)$.

Figure 1 also demonstrates that it is not sufficient for a point to lie on a maximal exposed face of U (as opposed to $\mathrm{dc}(U)$) to guarantee it maximizes a positive normal. Consider the point u'', which is a maximal exposed extreme point of U, but clearly does not maximize any positive normal over U. However, u'' does not lie on a maximal exposed face of $\mathrm{dc}(U)$ and so does not contradict the theorem.

Given the above nuance, a simpler sufficient condition may be useful. Consider the setting where all maximal faces of $\mathrm{dc}(U)$ are exposed.

Corollary 4. If U is a closed convex set such that all maximal faces of $\mathrm{dc}(U)$ are exposed, then $U^P = U^{++}$.[15]

Proof. Note that every maximal element of U lies in a maximal face of $\mathrm{dc}(U)$ by Lemma 4. This and the hypothesis imply that every maximal element of U belongs to some exposed maximal face of $\mathrm{dc}(U)$. Applying Theorem 3, we obtain the desired conclusion. □

However, the converse of Corollary 4 is false, as illustrated by the example in Figure 6. One sufficient condition for the hypothesis of Corollary 4 to hold is that U is a polyhedron. In that case, all faces of U are all exposed (Theorem 13.21 of Soltan (2015)); moreover, its downward closure of a polyhedron is also a polyhedron (Theorem 13.20 of Soltan (2015)), so all of its faces are exposed. Utility possibility sets that arise as polyhedra is not an

[15] This cannot be derived easily from Arrow, Barankin, and Blackwell (1953). To see this, recall that they establish $U^{++} \subset U^P \subset \mathrm{cl}(U^{++})$. This implies that if U^{++} is closed then $U^P = U^{++}$. However, in the "tilted cone" in Figure 2, U^{++} is not closed since the point K does not lie in U^{++} but is the limit point of elements in U^{++} (indicated by the line in the figure). However, it is straightforward to check that U^P and U^{++} coincide. One can also check that all maximal faces of $\mathrm{dc}(U)$ for U in Figure 2 are exposed, the condition of Corollary 4.

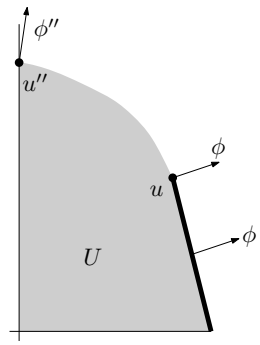


Figure 6: The maximal extreme point u is not exposed while $U^P = U^{++}$.

uncommon phenomenon. For example, the following special class of utility functions gives rise to such a case.

Let X be a polyhedral subset of $\mathbb{R}^m_+$ (possibly $\mathbb{R}^m_+$ itself). The utility function $u_i : X \to \mathbb{R}$ is **piecewise-linear concave (PLC)** if there exist finite index set K_i and affine functions $u_{i,k} : \mathbb{R}^m_+ \to \mathbb{R}$ for each $k \in K_i$ such that $u_i(x) = \min_{k \in K_i} u_{i,k}(x)$ for all $x \in X$.

PLC utility functions have appeared elsewhere in the literature. For instance, Afriat's theorem (Afriat (1967)) shows that they arise naturally in the context of revealed preferences. Also, the PLC case features prominently in the computer science literature on questions of hardness in computing equilibria (see Chen, Dai, Du, and Teng (2009) and Garg, Mehta, Vazirani, and Yazdanbod (2017) for instance). Moreover, it is well-known that concave functions can be approximated arbitrarily well by PLC functions with sufficiently many pieces (see, for instance, Bronshteyn and Ivanov (1975); Ghosh, Pananjady, Guntuboyina, and Ramchandran (2019)).

Proposition 1. If each agent has a PLC utility function defined on a polyhedron X and U is defined according to (1), then $\mathrm{dc}(U)$ is a polyhedron.

The following is obtained immediately from Corollary 4 and Proposition 1, and the fact that all faces of polyhedra are exposed. It is a clean economic setting where U^P and U^{++} coincide.

Theorem 4. If each agent has a PLC utility function defined on a polyhedron X and U is defined according to (1), then $U^P = U^{++}$.[16]

[16]It is worth noting that the ABB theorem provides an alternative proof of this result. Recall that it suffices to argue U^{++} is closed in order to conclude $U^P = U^{++}$. Clearly, the elements of U^{++} comes in faces, and a polyhedron has finitely-many faces. Since faces of a polyhedron are closed, and a finite union of closed sets is closed, this implies that U^{++} is closed.

5 Second Welfare Theorem with Piecewise Linear Concave Utility Functions

The notions of exposed face and normal vector play crucial roles for our characterization of a Pareto optimal utility profile as a welfare-maximizing point. Recall that the normal vector also plays an important role in the second theorem of welfare economics in identifying a price vector that supports a Pareto optimal allocation as a competitive equilibrium outcome. Unlike in our characterization, the idea of a normal vector in the second welfare theorem applies to the space of goods, not the space of utility profiles. However, the fact that the two spaces are closely connected hints at the possibility of establishing the second welfare theorem using the machinery we have developed so far. We do so in the current section under a set of assumptions on the agent preferences and endowments that generalize the existing welfare theorem in a certain direction.

To begin, consider an exchange economy with n agents (index by i) and m goods (indexed by k) introduced in Section 4.1. Suppose that each agent i is endowed with a vector of goods $e_i \in \mathbb{R}^m_+ \backslash \{0\}$ and let $\bar{e} = \sum_{i \in I} e_i$. A vector $p \in \mathbb{R}^m$ is referred to as a price profile. A pair (p, x) of a price profile p and a profile $x = (x_i)_{i \in I}$ of consumption bundles is a *Walrasian equilibrium* if

1. $\sum_{i \in I} x_i = \bar{e}$, and
2. $x_i \in \arg\max_{y_i \in B_i(p)} u_i(y_i)$ for each $i \in I$, where $B_i(p) := \{y_i \in \mathbb{R}^m_+ \mid \langle p, y_i \rangle \leq \langle p, e_i \rangle\}$ is the budget set of i.

We consider a case where utility functions of all players are piecewise-linear concave (PLC), as defined in Section 4.2. PLC utility functions may appear somewhat restrictive, but as noted earlier any concave function can be approximated arbitrarily closely by a PLC utility function. Meanwhile, we make a weaker assumption in another dimension—preference monotonicity. The existing second welfare theorem assumes agents' utility functions to be strictly monotonic. We invoke a weaker form of monotonicity. Say that an allocation $(x_i)_{i \in I}$ is *strictly feasible for good* k if it is feasible and satisfies $\sum_{i \in I} x_i^k < \bar{e}^k$. We assume that the agent preferences are **monotonic under limited resources** in the following sense: for any allocation $(x_i)_{i \in I}$ that is strictly feasible for good k, there exist an agent j and $\tilde{x}_j \in \mathbb{R}^m_+$ such that $u_j(\tilde{x}_j) > u_j(x_j)$ while $\tilde{x}_j^{k'} = x_j^{k'}, \forall k' \neq k$, $\tilde{x}_j^k > x_j^k$, and $\tilde{x}_j^k + \sum_{i \neq j} x_j^k \leq \bar{e}^k$. That is, given any allocation that does not exhaust the endowment of good k, there exists an agent who gets better off by consuming more of that good within its endowment. This condition is fairly weak. For instance, it allows for agents to consider certain good indifferently or even as bads (rather than goods), as long as there is at least one agent who likes to consume that good. We are now ready to prove the second welfare theorem under the above assumptions.

Theorem 5. Consider the exchange economy described above. If $(u_i(e_i))_{i \in I}$ is Pareto optimal, then there exists a positive price vector $p \gg 0$ such that $(p, (e_i)_{i \in I})$ is a Walrasian equilibrium.

In addition to the weakening of preference monotonicity, we also dispense with the typical assumption required by the existing second-welfare theorem that every consumer have a

positive endowment for every type of good (i.e., $e_i \gg 0, \forall i \in I$). The positive endowment assumption can be quite restrictive, excluding many realistic situations. In fact, relaxing the same assumption was an important motivation behind Arrow's generalization of the first welfare theorem.[17]

At the same time, the theorem assumes PLC utility functions. This assumption guarantees that the "upper contour set" of the target allocation—or the set of goods weakly preferred to $(e_i)_{i \in I}$—is a polyhedron. Meanwhile, preference monotonicity and Pareto-optimality of $(u_i(e_i))_{i \in I}$ ensure that the vector $\bar{e}$ is a (minimal) face of this set. Invoking Theorem 4, $\bar{e}$ is then exposed by a positive normal (or price vector) that supports $(e_i)_{i \in I}$ as a competitive equilibrium allocation.

6 Conclusion

In this paper, we characterized Pareto optimality via a new notion of *sequential* utilitarian welfare maximization. We established a strong connection between Pareto optimality and the geometric concept of exposed faces, where sequentiality was tied to the notion of "eventual" exposure. We used these insights to obtain conditions for characterizations by simpler, nonsequential utilitarian welfare maximizations and highlighted implications for polyhedral sets of utility vectors (and their associated PLC utility functions) whose faces are all exposed. This connection allowed us to establish a second welfare theorem in economies with PLC utilities under weaker regularity conditions than those in the existing literature.

The application of our methodology to the second welfare theorem suggests two related areas of exploration for future work. The first relates to further exploration of how our main results drive implications for problems stated in the choice space X, as opposed to the utility possibility space U. Indeed, examining the structure of what points in the choice set give rise to Pareto optima has been a major focus in the multi-objective optimization literature. An early contribution in that literature is Charnes and Cooper (1967), who showed an equivalence between the problem of finding Pareto-optimal solutions (in the choice set X) and that of solving a constrained nonlinear programming problem. Following their contribution, techniques in nonlinear programming were utilized to characterize Pareto optima under various conditions (Ehrgott, 2005; Ben-Israel, Ben-Tal, and Charnes, 1977; Van Rooyen, Zhou, and Zlobec, 1994; Glover, Jeyakumar, and Rubinov, 1999; Ben-Tal, 1980) all of which require some form of differentiability of the utility functions. We believe further investigation into our approach may have the potential to add to this literature in at least two aspects. First, our characterization does not assume any form of differentiability. Indeed, the subtlety involving non-exposure of maximal faces often arises when utility functions are not smooth (e.g., Example 1). Our methods may suggest ways to handle Pareto optimality when differentiability fails. Second, our methods may suggest a bridge between existing results in the domain space and results in the utility possibility space, where notions of (sequential)

[17] "While listening to a talk about housing by Franko Modigliani, Arrow realized that most people consume nothing of most goods (for example living in just one particular kind of house), and thus that the prevailing efficiency proofs assumed away all the realistic cases," according to Geanakoplos (2019).

welfare maximization are salient and allow for more natural economic interpretations. Indeed, none of the characterizations in the above references are in terms of notions of welfare maximization.

The second area of future work, partially inspired by the application of our methodology to the second welfare theorem, would be to examine how the notion of exposure can be used to enhance separating hyperplane arguments that may arise in other economic settings. Indeed, the standard proof of the second welfare theorem uses a supporting hyperplane argument at the endowment point but does not leverage the fact that this point is a minimal element of the "upper contour set." We show how this minimality allows us to strengthen usual separation arguments to guarantee the existence of strictly positive prices (as opposed to just nonnegative prices). This strict positivity allowed us to relax assumptions that are typically used to guarantee strictly positive wealth (e.g., strictly positive endowment for every agent). We believe there is scope to explore other economic settings where separating hyperplane arguments are used and similarly relax conditions needed to ensure strict positivity when maximality or minimality (in combination with notions of exposure) may be used to assure the existence of a separating hyperplane with a strictly positive normal.

A Proofs of preliminary results for Theorem 1

A.1 Proof of Lemma 2

The stated result is immediate in the case F is a singleton, so we may assume that F is not a singleton. Suppose for contradiction that F contains a nonmaximal element u'. Thus, there exists a $v \in U$ such that $v > u'$. Since $u \in \mathrm{ri}(F)$, there exists $\lambda > 0$ such that $w' = u + \lambda(u - u') \in F$. Now let $z = \alpha w' + (1-\alpha)v$, where $\alpha = \frac{1}{1+\lambda}$ or $\alpha(1+\lambda) = 1$. Note that $z \in U$ since U is convex. Also,

$$z = \alpha\left(u + \lambda(u - u')\right) + (1-\alpha)v = u - \alpha\lambda u' + (1-\alpha)v = u + (1-\alpha)(v - u') > u,$$

contradicting the maximality of u.

A.2 Proof of Lemma 4

Let U be a closed convex set and $\mathrm{dc}(U)$ its downward closure. Let u be a maximal element of $\mathrm{dc}(U)$; that is, $(u + \mathbb{R}^n_+) \cap \mathrm{dc}(U) = \{u\}$. If $u \in U$ then this implies $(u + \mathbb{R}^n_+) \cap U = \{u\}$ since $U \subset \mathrm{dc}(U)$ and so u is a maximal element of U. Note that if $u \in \mathrm{dc}(U) \setminus U$ then it cannot be maximal. Indeed, this implies that $u = v - w$ for some $v \in U$ and nonzero $w \in \mathbb{R}^n_+$ and so $v > u$ and so u is not maximal.

Conversely, we prove the contrapositive. Suppose $u \in \mathrm{dc}(U)$ is not a maximal element. This implies that there exists a $w \neq u$ with $w \in \mathrm{dc}(U)$ and $w \geq u$. But then we can find a $v \geq w \geq u$ and $v \neq u$ and $v \in U$. This implies that u is not a maximal element of U.

We next prove the second statement. To see that F is a face of $\mathrm{dc}(U)$, consider any $x, y \in \mathrm{dc}(U)$ and $\lambda \in (0,1)$ such that $z = \lambda x + (1-\lambda)y \in F$. We need to show that

both x and y belong to F. We first show that x and y are both maximal. Suppose for contradiction that x is not maximal. Then, we must have some $x' \in \mathrm{dc}(U)$ such that $x' > x$. Let $z' = \lambda x' + (1-\lambda)y$ and observe that $z' \in \mathrm{dc}(U)$, $z' \geq z$, and $z' \neq z$, which contradicts the maximality of z. Given that x and y are both maximal, we must have $x, y \in U$ since there is no maximal point in $\mathrm{dc}(U) \backslash U$. That F is a face of U then implies $x, y \in F$ as desired.

A.3 Proof of Lemma 5

Suppose there exists a supporting normal ϕ (i.e., there exists a β_ϕ such that $\langle \phi, u \rangle \leq \beta_\phi$ for all u in U) with a negative component ϕ_j for some $j \in J$. Let v be an arbitrary element of U. Since U is downward closed in coordinates J, we also have $v - \lambda \chi^j \in U$ for any $\lambda \geq 0$, where χ^j is the unit vector with 1 in component j. However, observe that $\langle \phi, v - \lambda \chi^j \rangle = \langle \phi, v \rangle - \lambda \langle \phi, \chi^j \rangle = \langle \phi, v \rangle - \lambda \phi_j$. But $\langle \phi, v \rangle - \lambda \phi_j \to \infty$ as $\lambda \to \infty$ since $\phi_j < 0$. This contradicts the fact that ϕ is a supporting normal.

A.4 Proof of Lemma 6

Take any $j \in K := J \setminus \operatorname{supp} \phi$ and set $u' = u - \epsilon \chi^j$ for some $u \in F$ and $\epsilon > 0$. Since U is downward closed in coordinates J and $j \in J$, we have $u' \in U$. Moreover, $\langle \phi, u' \rangle = \langle \phi, u - \epsilon \chi^j \rangle = \langle \phi, u \rangle - \epsilon \langle \phi, \chi^j \rangle = \langle \phi, u \rangle - \epsilon \phi_j = \langle \phi, u \rangle$ since $\phi_j = 0$ when $j \in K$ since no element of K lies in $\operatorname{supp} \phi$. But then $u' \in F$ since $\langle \phi, u' \rangle = \langle \phi, u \rangle = \max_{v \in U} \langle \phi, v \rangle$ and $F = \max_{v \in U} \langle \phi, v \rangle$ since F is exposed by ϕ.

B Proof of Theorem 2

Proof. The relationship $U^P \subset U^+$ follows from Theorem 1 because if u is Pareto optimal, then it maximizes a sequential set of normals $\Phi = (\phi^1, \ldots, \phi^T)$, and hence u maximizes a nonnegative normal ϕ^1. In the remainder of this proof, we will show the relation $U^+ \subset U^P$.

Suppose for contradiction that the desired conclusion does not hold. Then there exists $u \in U^+ \setminus U^P$. More specifically, there exists some $u' \in U$ such that $u' > u$ while $u \in \arg\max_{v \in U} \langle \phi, v \rangle$ for some nonnegative normal ϕ. We first note that $\phi_j = 0$ for every $j \in I$ such that $u'_j > u_j$. This is because otherwise $\phi_j > 0$ and $u'_j > u_j$, but this and the fact that $u' \geq u$ imply $\langle \phi, u' \rangle > \langle \phi, u \rangle$, contradicting the assumption that $u \in \arg\max_{v \in U} \langle \phi, v \rangle$. Now, fix $j \in I$ with $\phi_j = 0$ and $u'_j > u_j$: Note that there exists such $j \in I$ because $u' > u$. Also, fix $j' \in I$ such that $\phi_{j'} > 0$; note that such j' exists because ϕ is a nonnegative normal and that $j' \neq j$ because $\phi_j = 0$. Then, because $u_j \geq 0$ (recall that the minimum utility for each individual is normalized to 0), we have $u'_j > u_j \geq 0$. Let $x = (x_i)_{i \in I} \in X$ be such that $u' \leq (u_i(x_i))_{i \in I}$: Such x exists by the definition of U and the assumption that $u' \in U$. Then it follows that $u_j(x_j) \geq u'_j > 0$, so $x_j \geq 0$ and $x_j \neq 0$ because $x_j \in \mathbb{R}^m_+$ and $u_j(0) = 0$. Then consider an alternative consumption profile $y \in \mathbb{R}^m_+$ defined as $y_j = 0, y_{j'} = x_{j'} + x_j$, and $x_i = y_i$ for all $i \neq j, j'$; Note that $y \in X$ (this is because $\sum_{i \in I} y_i = \sum_{i \in I} x_i$ by definition of y and $\sum_{i \in I} x_i \leq \bar{e}$ by the assumption that $x \in X$) and hence $u(y) \in U$. Then, because the

utility function $u_{j'}(\cdot)$ is strictly monotonic by assumption while $x_j \geq 0$ and $x_j \neq 0$, we have $u_{j'}(y_{j'}) > u_{j'}(x_{j'}) \geq u'_{j'} \geq u_{j'}$. Moreover, by construction we have $u_i(y_i) = u_i(x_i) \geq u'_i \geq u_i$ for every $i \neq j, j'$. Therefore, because $\phi_j = 0$ and $\phi_{j'} > 0$, we have $\langle \phi, u(y) \rangle > \langle \phi, u \rangle$, which is a contradiction to the assumption that $u \in \arg\max_{v \in U} \langle \phi, v \rangle$. □

C Proof of Theorem 3

The proof uses results from the following three lemmas:

Lemma C.1 (Line Segment Principle, see Proposition 1.3.1 in Bertsekas (2009))**.** Let U be a closed convex set. If $u \in \mathrm{ri}(U)$ and $v \in U$, then $[u, v) \in \mathrm{ri}(U)$, where $[u, v) := \{u' \in U | u' = \lambda u + (1 - \lambda)v, \exists \lambda \in (0, 1]\}$.

The *normal cone* of U at a point $u \in U$ is the set

$$N_U(u) = \{\phi \in \mathbb{R}^n \mid \langle \phi, u \rangle \geq \langle \phi, v \rangle \text{ for all } v \in U\}.$$

If $\phi \in N_U(u)$ then u is a maximizer of the linear function $\langle \phi, u \rangle$ over the set U.

Lemma C.2. Let F be a face of a convex set U. Then every point in the relative interior of F has the same normal cone.

Proof. Let u, u' be distinct in the relative interior of F and suppose $N_U(u)$ contains an element ϕ not in $N_U(u')$. This implies $\langle \phi, u \rangle > \langle \phi, u' \rangle$. Since u is the relative interior, the point $v = u + \lambda(u - u')$ lies in F for a sufficiently small positive λ. But, $\langle \phi, v \rangle = \langle \phi, u \rangle + \lambda \langle \phi, u - u' \rangle > \langle \phi, u \rangle$, violating the assumption that ϕ is in $N_U(u)$. □

The above result lets us define the normal cone of a face F of U, denoted $N_U(F)$, as the normal cone of each of its relative interior points.Next, let us consider the relative boundary of F, defined as $F \setminus \mathrm{ri}(F)$. As the next result shows, the relative boundary points of a face F must contain $N_U(F)$ and additional normal vectors.

Lemma C.3. Let F be a face of a convex set U. Then every relative boundary point u of F has $N_U(u) \supset N_U(F)$.

Proof. Let u be in the relative boundary of F. Suppose there is a normal ϕ in $N_U(v)$ (where v is any relative interior element of F) that is not in $N_U(u)$. That is,

$$\langle \phi, u \rangle \neq \langle \phi, v \rangle. \tag{8}$$

By the Line Segment Principle, we can get an element of relative interior of F arbitrarily close to u, which yield a contradiction of the continuity of $\langle \phi, \cdot \rangle$ because of (8). □

Proof of Theorem 3. ($\Leftarrow$) Observe that $U^{++} \subset U^P$ is immediate from Proposition 16.E.2 in Mas-Colell, Whinston, and Green (1995). It remains to show $U^P \subset U^{++}$. Let $u \in U^P$. If u lies in the relative interior of an exposed face of $\mathrm{dc}(U)$, then $u \in U^{++}$ from Corollary 3.

The remaining case is where u lies on the relative boundary of a maximal exposed face F of $\mathrm{dc}(U)$. Since F is a maximal exposed face, then an element v in its relative interior maximizes a positive normal ϕ, again by Corollary 3. By Lemma C.2, this implies that the normal cone $N_U(F)$ of face F contains ϕ and so, by Lemma C.3, the normal cone $N_U(u)$ of the point u contains ϕ. In other words, u maximizes the positive normal ϕ. This completes the proof.

($\Rightarrow$) Let u be a maximal element of U. By the equivalence of U^P and U^{++}, u maximizes a positive normal ϕ. Let $F = \arg\max_{v \in U} \langle \phi, v \rangle$. We claim that F is a maximal exposed face of $\mathrm{dc}(U)$, which clearly contains u. The fact that F is maximal in $\mathrm{dc}(U)$ follows since Proposition 16.E.2 in Mas-Colell, Whinston, and Green (1995) (along with Lemma 2) implies F is maximal in U and thus maximal in $\mathrm{dc}(U)$ by Lemma 4. Suppose to the contrary that F is not exposed in $\mathrm{dc}(U)$. Then, there must exist an element $u' \in \mathrm{dc}(U) \setminus U$ that maximizes ϕ but is not in F. However, since u' in $\mathrm{dc}(U) \setminus U$ there must exist a $u'' \in U$ such that $u' \leq u''$ and $u'_i < u''_i$ for some index i. But, this implies that $\langle \phi, u \rangle \geq \langle \phi, u'' \rangle > \langle \phi, u' \rangle$, where the weak inequality holds by the definition of F and the strict inequality holds since ϕ is positive. This yields a contradiction and so we conclude that F is an exposed face of $\mathrm{dc}(U)$. $\square$

D Proof of Proposition 1

For each $k \in K_i$, let $X_{i,k} = \{x \in X \mid u_{i,k}(x) \leq u_{i,k'}(x), \forall k' \in K_i\}$. Since X is a polyhedron and all functions $(u_{i,k})_{k \in K_i}$ are affine, $X_{i,k}$ is an intersection of finitely many polyhedra and thus a polyhedron.

Now let $\mathcal{K} = \{\mathbf{k} = (k_i)_{i \in I} \mid k_i \in K_i \text{ for all } i\}$. For each $\mathbf{k} \in \mathcal{K}$, let $X_{\mathbf{k}} = \cap_{i \in I} X_{i,k_i}$ and observe that $X_{\mathbf{k}}$ is a polyhedron. Also, all functions $u_1(\cdot), \ldots, u_I(\cdot)$ are affine on $X_{\mathbf{k}}$ since for each $i \in I$, $u_i(x) = u_{i,k_i}(x), \forall x \in X_{\mathbf{k}}$. Then, by Theorem 13.21 of Soltan (2015), the set $U_{\mathbf{k}} = \{(u_i(x))_{i \in I} \mid x \in X_{\mathbf{k}}\}$ is a polyhedron. Observe that $U = \{(u_i(x))_{i \in I} \mid x \in X\} = \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}}$. While we do not know whether the set U, which is a union of polyhedra, is a polyhedron, Theorem 13.19 of Soltan (2015) shows that $\overline{U} := \mathrm{cl}(\mathrm{conv} \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}})$ is a polyhedron, where cl and conv denote the closure and convex hull, respectively.

Next, we show that $\mathrm{dc}(U) = \mathrm{dc}(\overline{U})$. By definition of $\overline{U}$, $\mathrm{dc}(U) \subset \mathrm{dc}(\overline{U})$ is clear. To show $\mathrm{dc}(\overline{U}) \subset \mathrm{dc}(U)$, consider any $\tilde{u} \in \mathrm{conv} \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}}$ so that $\tilde{u} = \sum_{\mathbf{k} \in \mathcal{K}} \lambda_{\mathbf{k}} \tilde{u}_{\mathbf{k}}$ for some weight $(\lambda_{\mathbf{k}})_{\mathbf{k} \in \mathcal{K}}$ and $\tilde{u}_{\mathbf{k}} \in \cup_{\mathbf{k}' \in \mathcal{K}} U_{\mathbf{k}'}$. Also, for each $\tilde{u}_{\mathbf{k}}$, we can find $\tilde{x}_{\mathbf{k}} \in X_{\mathbf{k}}$ such that $(u_i(\tilde{x}_{\mathbf{k}}))_{i \in I} = \tilde{u}_{\mathbf{k}}$. Letting $x = \sum_{\mathbf{k} \in \mathcal{K}} \lambda_{\mathbf{k}} \tilde{x}_{\mathbf{k}}$, observe that $x \in X$ by the convexity of X and that for all $i \in I$, $u_i(x) \geq \sum_{\mathbf{k} \in \mathcal{K}} \lambda_{\mathbf{k}} u_i(\tilde{x}_{\mathbf{k}}) = \tilde{u}_i$ by the concavity of $u_i(\cdot)$, which means that $\tilde{u} \in \mathrm{dc}(U)$. Thus, $\mathrm{conv} \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}} \subset \mathrm{dc}(U)$, implying that $\mathrm{cl}(\mathrm{conv} \cup_{\mathbf{k} \in \mathcal{K}} U_{\mathbf{k}}) \subset \mathrm{dc}(U)$ since $\mathrm{dc}(U)$ is closed, from which $\mathrm{dc}(\overline{U}) \subset \mathrm{dc}(U)$ follows, as desired.

Lastly, observe that $\mathrm{dc}(\overline{U}) = \overline{U} + \mathbb{R}^n_-$ and that both $\overline{U}$ and $\mathbb{R}^n_-$ are polyhedra, which implies (by Theorem 13.20 of Soltan (2015)) that $\mathrm{dc}(\overline{U}) = \mathrm{dc}(U)$ is a polyhedron.

E Proof of Theorem 5

The proof of Theorem 5 uses the following preliminary results.

Lemma E.1. The following properties on polyhedra hold:

(i) Let $P_1, P_2, \ldots, P_n$ be a finite collection of polyhedra in $\mathbb{R}^m$. The Cartesian product $P_1 \times P_2 \times \cdots \times P_n$ is a polyhedron in $\mathbb{R}^{mn}$.

(ii) Let $\pi : \mathbb{R}^d \to \mathbb{R}^m$ be a affine map and let P be a polyhedron in $\mathbb{R}^d$. Then $\pi(P)$ is a polyhedron.

(iii) All faces of a polyhedron are exposed.

(iv) The downward closure of a polyhedron is also a polyhedron.

Proof. (i) Consider two polyhedra in $\mathbb{R}^m$, P_1 and P_2. Letting $Q_1 := P_1 \times \mathbb{R}^m$ and $Q_2 := \mathbb{R}^m \times P_2$, each Q_k is a polyhedron in $\mathbb{R}^{2m}$, so $P_1 \times P_2 = \cap_{k=1,2} Q_k$ is a polyhedron in $\mathbb{R}^{2m}$. The result follows from applying this argument repeatedly. (ii) This is Theorem 13.21 in Soltan (2015). (iii) This is Corollary 13.12 in Soltan (2015). (iv) This follows since $\mathrm{dc}(P) = P + \mathbb{R}^n_-$ where $\mathbb{R}^n_-$ is the nonpositive orthant of $\mathbb{R}^n$ and applying Theorem 13.20 of Soltan (2015). □

Let $A_i := \{x \in \mathbb{R}^m_+ \mid u_i(x) \geq u_i(e_i)\}$ for each agent i. Observe that each A_i is a polyhedron since it is an intersection of two polyhedra, $\{x \in \mathbb{R}^m \mid x \geq 0\}$ and $\{x \in \mathbb{R}^m \mid u_i(x) \geq u_i(e_i)\} = \cap_{k \in K_i} \{x \in \mathbb{R}^m \mid u_{i,k}(x) \geq u_i(e_i)\}$

Consider the set $A = \left\{x \in \mathbb{R}^m_+ \mid \exists x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n \text{ s.t. } x = \sum_{i \in I} x_i\right\}$. Observe that A is the image of the set $A_1 \times A_2 \times \cdots \times A_n$ under the affine mapping π that maps $(x_i)_{i \in I}$ to $\sum_{i \in I} x_i$. By Lemma E.1(i) and (ii), A itself is a polyhedron.

Next, we argue that $\bar{e}$ is a minimal element of the set A. Suppose for contradiction that there exists an element $x \in A$ where $x < \bar{e}$ where $x^k < \bar{e}^k$ for some good k. Since $x \in A$, there exists an allocation $(y_i)_{i \in I}$ where $y_i \in A_i$ such that $x = \sum_{i \in I} y_i$. Since this allocation is strictly feasible for the good k, the monotone preference under limited resources implies that there are some agent j and $\tilde{y}_j \in \mathbb{R}^m_+$ such that $u_j(y_j) < u_j(\tilde{y}_j)$ while $\tilde{y}_j^{k'} = y_j^{k'}, \forall k' \neq k$, $\tilde{y}_j^k > y_j^k$, and $\tilde{y}_j^k + \sum_{i \neq j} y_i^k \leq \bar{e}^k$. Now consider an alternative allocation $(z_i)_{i \in I}$, which is identical to $(y_i)_{i \in I}$ except that $z_j = \tilde{y}_j$. Note that this allocation is feasible under the endowment $\bar{e}$ and that $u_j(z_j) > u_j(y_j) \geq u_j(e_j)$ while $u_i(z_i) = u_i(y_i) \geq u_i(e_i), \forall i \neq j$, which contradicts the Pareto optimality of $(e_i)_{i \in I}$.

That $\bar{e}$ is a minimal element of A implies that $-\bar{e}$ is a maximal element of $-A$. By Lemma 4, this implies that $-\bar{e}$ is a maximal element of $\mathrm{dc}(-A)$. Moreover, by Lemma E.1(iv) $\mathrm{dc}(-A)$ is a polyhedron and so by Lemma E.1(iii) all of its faces are exposed. Thus, by Corollary 3, there exists a supporting hyperplane of $-A$ through the point $-\bar{e}$ with a positive normal ϕ. The same normal $p := \phi$ can define a supporting hyperplane to A through the point $\bar{e}$; that is,

$$\langle p, y \rangle \geq \langle p, \bar{e} \rangle, \forall y \in A,$$

where p is a strictly positive vector of prices.

It remains to show that the positive price vector p just constructed supports the allocation $(e_i)_{i \in I}$ as a Walrasian equilibrium. For this, it suffices to show that each e_i maximizes $u_i(\cdot)$

under the prices p and the budget $\langle p, e_i \rangle$. To do so, we take any x_i with $u_i(x_i) > u_i(e_i)$ and will show that agent i cannot afford x_i.

By continuity of u_i, the inequality $u_i(x_i) > u_i(e_i)$ implies that for some $\lambda < 1$ but sufficiently close to 1, we have $u_i(\lambda x_i) > u_i(e_i)$, so by definition we have $\lambda x_i \in A_i$. This implies that $\lambda x_i + \sum_{j \neq i} e_j \in A$. Since $\langle p, \lambda x_i + \sum_{j \neq i} e_j \rangle \geq \langle p, \sum_{i \in I} e_i \rangle$, we must also have $\langle p, \lambda x_i \rangle \geq \langle p, e_i \rangle$. Dividing through by λ gives $\langle p, x_i \rangle \geq \langle \frac{1}{\lambda} p, e_i \rangle > \langle p, e_i \rangle$ where the strict inequality holds since e_i is nonnegative and nonzero while p is strictly positive.

References

Afriat, S. N. (1967): "The construction of utility functions from expenditure data," *International Economic Review*, 8(1), 67–77. 6, 18

Argenziano, R., and I. Gilboa (2015): "Weighted utilitarianism, Edgeworth, and the market," Discussion paper, Mimeo. 16

Arrow, K., E. Barankin, and D. Blackwell (1953): "Admissible points of convex sets," in *Contributions to the Theory of Games*, ed. by Kuhn, and Tucker, pp. 87–91. Princeton University Press, Princeton, NJ. 2, 3, 4, 17

Ben-Israel, A., A. Ben-Tal, and A. Charnes (1977): "Necessary and sufficient conditions for a Pareto optimum in convex programming," *Econometrica*, pp. 811–820. 20

Ben-Tal, A. (1980): "Characterization of Pareto and lexicographic optimal solutions," in *Multiple Criteria Decision Making Theory and Application*, pp. 1–11. Springer. 20

Bertsekas, D. P. (2009): *Convex Optimization Theory*. Athena Scientific. 23

Bitran, G. R., and T. L. Magnanti (1979): "The structure of admissible points with respect to cone dominance," *Journal of Optimization Theory and Applications*, 29(4), 573–614. 3

Bronshteyn, E. M., and L. Ivanov (1975): "The approximation of convex sets by polyhedra," *Siberian Mathematical Journal*, 16(5), 852–853. 18

Charnes, A., and W. W. Cooper (1967): *Management Models and Industrial Applications of Linear Programming*. Wiley. 20

Che, Y.-K., J. Kim, and F. Kojima (2019): "Weak Monotone Comparative Statics," arXiv:1911.06442 [econ.TH]. 5, 8

Chen, X., D. Dai, Y. Du, and S.-H. Teng (2009): "Settling the complexity of Arrow-Debreu equilibria in markets with additively separable utilities," in *IEEE Symposium on Foundations of Computer Science (FOCS)*, pp. 273–282. IEEE. 18

Daniilidis, A. (2000): "Arrow-Barankin-Blackwell theorems and related results in cone duality: A survey," in *Optimization*, pp. 119–131. Springer. 2

Ehrgott, M. (2005): *Multicriteria Optimization*. Springer. 20

Garg, J., R. Mehta, V. V. Vazirani, and S. Yazdanbod (2017): "Settling the complexity of Leontief and PLC exchange markets under exact and approximate equilibria," in *ACM Symposium on Theory of Computing (STOC)*, pp. 890–901. 18

Geanakoplos, J. (2019): "On being a student of Ken Arrow," *Quantitative Finance*, 19. 20

Ghosh, A., A. Pananjady, A. Guntuboyina, and K. Ramchandran (2019): "Max-Affine Regression: Provable, Tractable, and Near-Optimal Statistical Estimation," *arXiv preprint arXiv:1906.09255*. 18

Glover, B. M., V. Jeyakumar, and A. M. Rubinov (1999): "Dual conditions characterizing optimality for convex multi-objective programs," *Mathematical Programming*, 84(1), 201. 20

Kalai, E. (1977): "Nonsymmetric Nash Solutions and Replications of 2-Person Bargaining," *International Journal of Game Theory*, 6, 129–133. 8

Lopomo, G., L. Rigotti, and C. Shannon (2019): "Detectability, Duality, and Surplus Extraction," arXiv:1905.12788 [econ.TH]. 12

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. Oxford University Press. 2, 4, 16, 23, 24

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180. 5

Soltan, V. (2015): *Lectures on Convex Sets*. World Scientific. 6, 11, 12, 13, 17, 24, 25

Svensson, L.-G. (1999): "Strategy-proof allocation of indivisible goods," *Social Choice and Welfare*, 16, 557–567. 5

Topkis, D. M. (1998): *Supermodularity and Complementarity*. Princeton University, Princeton, New Jersey. 5

Van Rooyen, M., X. Zhou, and S. Zlobec (1994): "A saddle-point characterization of Pareto optima," *Mathematical Programming*, 67(1-3), 77–88. 20

Yaari, M. E. (1981): "Rawls, Edgeworth, Shapley, Nash: Theories of distributive justice re-examined," *Journal of Economic Theory*, 24, 1–39. 5