Matching and Prices*

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Abstract

Indivisibilities and budget constraints are pervasive features of many matching markets. But gross substitutability—a standard condition on preferences in matching models—typically fails in such markets. To accommodate budget constraints and other income effects, we instead assume that agents' preferences satisfy net substitutability. Although competitive equilibria do not generally exist in our setting, we show that stable outcomes always exist and are efficient. We illustrate how the flexibility of prices is critical for our results. We also discuss how budget constraints and other income effects affect the properties of standard auction and matching procedures, as well as of the set of stable outcomes.

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1 Introduction

Many markets, such as online platforms, labor markets, and auctions, involve highly heterogeneous and indivisible transactions. In these markets, sellers are usually constrained in what they can sell, and buyers constrained in how much they can pay. For example, in large spectrum auctions, governments face constraints on the combinations of blocks of spectrum they can sell, and telecom companies have limited budgets for the purchase of spectrum (Milgrom, 2000; Bulow et al., 2017). Similarly, in labor markets, workers are limited in how many (or which) jobs they can have, while firms may have hiring budgets. However, it is well-known that competitive equilibria do not generally exist in markets with indivisibilities and budget constraints.

In this paper, we show how insights from matching theory can help analyze markets with indivisibilities and budget constraints, and illuminate the role of flexible prices in coordinating these markets. We model market interactions as a two-sided, many-to-many matching market with monetary transfers. The key assumption in our analysis is that agents view the goods that are traded in the market as *net substitutes*. This condition requires, for example, that if the price of a good rises, then buyers' Hicksian (viz. compensated) demands for all other goods weakly increase (Baldwin et al., 2020). We represent market outcomes as sets of *contracts* (in the spirit of Hatfield and Milgrom (2005)), each of which consists of a *trade* between agents and a price for that trade (Hatfield et al., 2013; Fleiner et al., 2019).[1] We then show that (under net substitutability) there always exist outcomes that are *stable* in the sense that they are not *blocked* by any set of contracts.[2]

Our existence result is particularly striking because stable outcomes in our model share few of the familiar properties from previous matching and auction analyses. This difference occurs because much of the previous work instead assumed that agents' preferences satisfy the *gross substitutability* condition—i.e., that if the price of a good rises, then buyers' Marshallian (viz. uncompensated) demands for other goods weakly increase (Kelso and Crawford, 1982). In that case, competitive equilibrium outcomes exist, are stable, and can be constructed using versions of standard auction and matching procedures, such as Gale and Shapley's (1962) Deferred Acceptance

[1] Trades could represent, for example, the sale of the good (Gul and Stacchetti, 1999), or the non-pecuniary aspects of a job contract (Crawford and Knoer, 1981; Kelso and Crawford, 1982).

[2] This definition of stability is due to Roth (1984) and Hatfield and Milgrom (2005).

algorithm.[3]

However, gross substitutability is difficult to reconcile with the presence of budget constraints or other income effects—even when a buyer can demand only two goods. For example, suppose that a firm values workers at \$5 each, and has a hiring budget of \$4. In this case, if two workers' salaries were \$1 and \$3, respectively, then the firm would hire both workers. But if the first worker's salary went up to \$2, then the firm would no longer wish to hire the second worker—a gross complementarity. However, the firm sees workers as net substitutes.[4] More generally, it turns out that net substitutability is a strictly weaker condition on preferences than gross substitutability.[5] In particular, net substitutability allows not only for gross substitutabilities, but also for forms of gross complementarities.

Due to possibility of gross complementarities in our model, we cannot apply constructive arguments to establish the existence of stable outcomes using standard procedures such as ascending auctions (Gul and Stacchetti, 2000), the Deferred Acceptance algorithm, the descending salary adjustment process (Kelso and Crawford, 1982), and the Cumulative Offer process (Hatfield and Milgrom, 2005). Indeed, we show that those procedures sometimes converge to unstable outcomes in our setting, even though stable outcomes exist. Our argument for existence instead combines methods from matching theory with techniques from general equilibrium theory. We adapt Debreu's (1962) notion of *quasiequilibrium* to our model, and show that quasiequilibria always exist by leveraging topological fixed-point arguments developed by Baldwin et al. (2020). We complete the argument by showing that quasiequilibria give rise to stable outcomes.

We then explore the role of price flexibility in our model. First, we show that price flexibility is critical for existence. Under gross substitutability, it is known that stable outcomes exist regardless of whether prices are flexible.[6] However, we show that under net substitutability, stable outcomes do not generally exist with rigid

[3]See Fleiner et al. (2019) for the most general versions of these results.

[4]For example, for the price change considered above, note that when the firm is fully compensated for the budgetary impact of the salary increase, it would continue to demand both workers.

[5]The relationship between gross and net substitutability with income effects, but without hard budget constraints was shown by Baldwin et al. (2020); this paper shows that the relationship carries over to settings with hard budget constraints.

[6]For the flexible price case, see, for example, Kelso and Crawford (1982), Hatfield et al. (2013), and Fleiner et al. (2019). For the rigid price case, see, for example, Roth (1984), Ostrovsky (2008), Hatfield and Milgrom (2005), and Hatfield and Kominers (2017).

prices. Indeed, net substitutable preferences do not even always satisfy a condition that Hatfield and Kojima (2008) showed to be necessary (in a maximal domain sense) for the existence of stable outcomes in many-to-one matching markets without flexible prices.[7] Thus, our results show that flexible prices play a key role in coordinating matching markets under net substitutability—as is familiar from typical equilibrium models with divisible goods.

Second, we explore how price flexibility affects the efficiency properties of stable outcomes. In many-to-many matching markets without flexible prices, stable outcomes can be outside the core and strictly Pareto-dominated—even under gross substitutability (Blair, 1988). We show that in our model, stable outcomes are always in the core. Hence, price flexibility can improve the efficiency of stable outcomes—even in the presence of gross complementarities and budget constraints.

Third, we show that price flexibility plays an important role in the ability of agents to focus on simple potential blocks. To achieve stability, agents would *a priori* have to consider arbitrarily complicated blocking sets. Under gross substitutability, however, any bilateral contract that is part of a blocking set is a profitable deviation on its own (Hatfield and Kominers, 2017). In particular, if an outcome is unstable, then there exists a block consisting of just a single contract. Surprisingly, this characterization of stability carries over to our setting despite the possibility of gross complementarities. However, given a blocking set of contracts, it is possible that no contract in the set forms a block on its own; instead, we show that there is a blocking contract that corresponds to the same trade as a contract in the original set, but with a different price. Hence, without price flexibility, agents would not be able to restrict their attention to pairwise blocks to in order to block an unstable outcome.

Finally, we show that other classic properties of stable outcomes fail in our setting. Under gross substitutability, it is known that the set of stable outcomes forms a lattice. Moreover, when a further condition on preferences known as the "Law of Aggregate Demand" (Hatfield and Milgrom, 2005) is also satisfied, any agent that is matched in one stable outcome is matched in every other stable outcome, and there are stable matching mechanisms that are strategy-proof for unit-demand agents on one side of

[7]Hatfield and Kojima's (2008) condition is called "weak substitutability," and is weaker than gross substitutability when there are multiple possible contracts between pairs of agents. In many-to-many matching markets without flexible prices, even gross substitutability is necessary (in a maximal domain sense) for the existence of stable outcomes (Hatfield and Kominers, 2017).

the market.[8] We show that none of these properties generally hold in our setting. In particular, the set of stable outcomes may not form a lattice,[9] agents may be unmatched in some stable outcomes but receive more than their autarky payoff in others, and even unit-demand agents can generally manipulate their reports to stable matching mechanisms to obtain better outcomes.

Related literature. There is a large literature on many-to-many matching markets that assumes gross substitutability.[10] That literature has analyzed various cooperative solution concepts without flexible prices (Roth, 1984; Blair, 1988; Alkan, 2002; Fleiner, 2003; Echenique and Oviedo, 2006; Klaus and Walzl, 2009; Hatfield and Kominers, 2017).[11] It has also connected those concepts to competitive equilibrium when there are continuous transfers (Hatfield et al., 2013; Fleiner et al., 2019). In our case, despite the presence of continuous prices, competitive equilibria fail to exist, but stable outcomes do not. Moreover, our analysis relies on a weaker condition on preferences than gross substitutability that is compatible with budget constraints.

Other papers have examined one-to-one matching markets with budget constraints. Herings and Zhou (2019) study a one-to-one model with financial constraints. In their setting, competitive equilibria can fail to exist, but stable outcomes have a lattice structure. A number of papers have considered dynamic auctions with hard budget constraints and unit-demand bidders (Talman and Yang, 2015; van der Laan and Yang, 2016; Zhou, 2017); in particular, Talman and Yang (2015) describe a core-selecting auction in that context. In our setting, by contrast, the structural properties of stable outcomes fail, and standard auction and matching procedures do not generally find stable outcomes.

Two recent papers have considered exchange economies with indivisible goods and income effects. Baldwin et al. (2020) showed that competitive equilibria exist under net substitutability in a setting without budget constraints. Nguyen and Vohra (2021) showed that competitive equilibria exist under generalization of Gul and Stacchetti's (1999) "single improvement property," which is closely related to net

[8]See Schlegel (2021) for the most general versions of these results.

[9]In fact, we show that there may not even be buyer- or seller-optimal stable outcomes.

[10]One exception is Rostek and Yoder (2020), who showed that stable outcomes exist when all agents view *all* contracts as (gross) complements—ruling out all forms of substitutability.

[11]These analyses have also been extended to matching in trading networks (Ostrovsky, 2008; Westkamp, 2010; Hatfield and Kominers, 2012).

substitutability.[12] While versions of Baldwin et al.'s (2020) arguments and methods underpin some of our analysis, neither Baldwin et al. (2020) nor Nguyen and Vohra (2021) incorporated the possibility of binding budget constraints or analyzed stable outcomes.

Outline of the paper. This paper is organized as follows. Section 2 sets up the model. Section 3 explains why competitive equilibria do not in general exist in our model. Section 4 states the main result on the existence of stable outcomes, and outlines the proof. Section 5 discusses the relationships between stability, the core, and pairwise stability. Section 6 gives examples of the failure of the lattice structure of stable outcomes, the conclusion of the Lone Wolf Theorem, and the existence of stable and strategy-proof mechanisms. Appendix A describes a special case of our model with income effects but without hard budget constraints. Appendix B develops a new relationship between Marshallian and Hicksian demands in a setting with indivisible goods and hard budget constraints which we use throughout the proofs. Appendix C presents the proofs. Appendix D provides additional details for some examples.

2 Model

Our model specializes the model of Hatfield et al. (2021) to two-sided matching markets, but relaxes their key assumption on preferences and explicitly incorporates endowments of money and budget constraints.

2.1 Agents and trades

There is a finite set B of buyers and a finite set S of sellers; we let $I = B \cup S$ denote the set of agents.

Agents interact via trades and payments of money. Formally, there is a finite set Ω of trades. Each trade $\omega \in \Omega$ is associated with a buyer $\mathsf{b}(\omega) \in B$ and a seller $\mathsf{s}(\omega) \in S$. For example, trade ω could represent the sale of a good from $\mathsf{s}(\omega)$ to $\mathsf{b}(\omega)$ (Gul and Stacchetti, 1999), or specify all the non-pecuniary aspects of a job contract between a worker $\mathsf{s}(\omega)$ and a firm $\mathsf{b}(\omega)$ (Crawford and Knoer, 1981; Kelso and Crawford, 1982).

[12] Nguyen and Vohra (2021) also extended this result to demonstrate the existence of approximate equilibrium under a weaker condition called "Δ-substitutability."

Given a set $\Xi \subseteq \Omega$ of trades and an agent $i \in I$, let Ξ_i denote the set of trades in Ξ in which i is involved (as either a buyer or a seller).

2.2 Preferences

Each agent's utility depends on the trades that involve the agent and on their consumption of money $m \in \mathbb{R}$.[13] Formally, agent i has a utility function

$$U^i : \mathcal{P}(\Omega_i) \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\},$$

where $\mathcal{P}$ denotes the power set operator. We place conditions on utility functions so the possibility that utility can take value $-\infty$ represents situations that violate technological constraints for sellers or budget constraints for buyers.[14]

Assumption 1 (Feasibility constraints for sellers)**.** For each seller s, there is a family $\mathcal{F}^s \ni \varnothing$ of sets of trades that are *feasible* for s such that $U^s(\Xi, m) \in \mathbb{R}$ for $\Xi \in \mathcal{F}^s$ and $U^s(\Xi, m) = -\infty$ for $\Xi \notin \mathcal{F}^s$.

Intuitively, Assumption 1 states that utility is finite for feasible sets of trades and any level of money, but no amount of money can make a seller willing to execute an infeasible set of trades. This structure captures the possibility of technological constraints for sellers; for example, a worker might not be able to work full-time at two jobs no matter what the salaries are; it also rules out budget constraints for sellers. Note that autarky is always required to be a feasible option for each seller—no seller is forced to participate in the market.

Assumption 2 (Feasibility constraints for buyers)**.** For each buyer b, there is a lower bound $\underline{m}^b \in \mathbb{R} \cup \{-\infty\}$ on the consumption of money such that $U^b(\Xi, m) \in \mathbb{R}$ for $m > \underline{m}^b$ and $U^b(\Xi, m) = -\infty$ for $m < \underline{m}^b$.[15]

Intuitively, Assumption 2 states that utility is finite as long as buyer b ends up with more than $\underline{m}^b$ amount of money, but that buyer b would never be willing to end up with less than $\underline{m}^b$ amount of money regardless of which trades are executed. As

[13]We therefore rule out the possibility of externalities and of distortionary frictions.

[14]The technological constraints that can arise here are similar to Hatfield et al. (2013) and Fleiner et al. (2019).

[15]For notation, we let $\underline{m}^s = -\infty$ for all sellers s since Assumption 1 rules out budget constraints for sellers.

the utility level $U^b\left(\Xi, \underline{m}^b\right)$ can be finite or $-\infty$, we allow for preferences for which the lower bound $\underline{m}^b$ can be achieved, and for preferences for which it cannot. Note that Assumption 2 rules out technological constraints for buyers—all sets of trades are feasible for buyers.

We also impose two more standard regularity conditions on utility functions.

Assumption 3 (Continuity)**.** All agents' utility functions are continuous in money away from level $-\infty$. Furthermore, for all buyers b, and sets $\Xi \subseteq \Omega_b$ of trades, we have that

$$\lim_{m \to (\underline{m}^b)^+} U^b(\Xi, m) = U^b\left(\Xi, \underline{m}^b\right), \tag{1}$$

where we write $U^b(\Xi, -\infty) = -\infty$.

The second part of Assumption 3 requires that buyers b's utility be right-continuous at $\underline{m}^b$. Intuitively, it states that utility levels at money amounts just above $\underline{m}^b$ must be close to the utility level at $\underline{m}^b$. In particular, this assumption rules out that utility "jump down" at money amount $\underline{m}^b$.

Assumption 4 (Monotonicity)**.** Away from utility level $-\infty$, all agents' utility functions are strictly increasing in money, and buyers' (resp. sellers') utility functions are weakly increasing (resp. weakly decreasing) in trades.

The second part of Assumption 4 is just a version of the free disposal condition.

Our final assumption is economically innocuous within the context of the model and only serves to simplify the proofs.[16]

Assumption 5 (Unboundedness)**.** For all sellers s, we have that

$$\lim_{m \to -\infty} U^s(\varnothing, m) = -\infty,$$

and for all buyers b, we have that

$$\lim_{m \to \infty} U^b(\varnothing, m) = \infty.$$

Assumption 5 says that utility gain from more money is unbounded for buyers and utility loss from too little money is unbounded for the sellers.[17]

We maintain Assumptions 1–5 throughout the paper.

[16]Footnote 42 in Appendix B explains the place in our arguments in which Asssumption 5 is used.
[17]As we discuss later, buyers start with finite incomes and cannot end up with more money than

2.2.1 Examples of preferences

We next introduce three economically important classes of preferences that satisfy Assumptions 1–5 in our model. In order to illustrate these classes, we consider valuations for trades in our context. A *valuation for i* is a function $V^i : \mathcal{P}(\Omega_i) \to \mathbb{R} \cup \{-\infty\}$ with $V^i(\varnothing) \in \mathbb{R}$. If i is a buyer, the valuation must be weakly increasing (and hence cannot take value $-\infty$); if i is a seller, the valuation must be weakly decreasing (and hence permitting some sets of trades to be valued at $-\infty$).

The following examples illustrate our three key classes of preferences. The first class is the standard quasilinear utility without a budget constraint.

Example 1 (Quasilinear utility without a budget constraint)*.* We have that

$$U^i(\Xi, m) = V^i(\Xi) + m$$

for some valuation V^i. In this case, $\underline{m}^i = -\infty$.

For convenience, we henceforth refer to quasilinear utility without a budget constraint simply as *quasilinear utility.* If an agent's utility function is quasilinear, it is sufficient to describe their valuations in order capture the agent's preferences.

The remaining two classes of preferences are defined for buyers, and feature budget constraints that can and cannot bind, respectively.

The second example of preferences involves buyers with quasilinear utility functions who additionally experience hard budget constraints.

Example 2 (Quasilinear utility with a hard budget constraint)*.* We have that

$$U^b(\Xi, m) = \begin{cases} V^b(\Xi) + m & \text{if } m \geq 0 \\ -\infty & \text{if } m < 0 \end{cases}$$

for some valuation V^b. In this case, $\underline{m}^b = 0$.

In Example 2, buyers behave as if they have quasilinear utility away from the hard budget constraint at $\underline{m}^b = 0$, but would be unwilling to end up with debt. Note that these preferences do not violate the Assumption 3 because utility "jumps down" just

their income because prices must be non-negative (by the monotonicity Assumption 4). For an analogous reason, sellers cannot make unbounded losses. Thus, Assumption 5 is technical rather than substantive.

beyond, rather than at, the budget constraint (i.e., utility is still right-continuous, but not left-continuous, at $m = 0$).

The third example covers utility functions that are not quasilinear but for which some money is essential; such utility functions are continuous. To give an example of such preferences, we use a convenient functional form developed by Baldwin et al. (2020).

Example 3 (Quasilogarithmic utility—Baldwin et al., 2020). We have that

$$U^b\left(\Xi, m\right) = \log m - \log(-V^b_{\mathrm{Q}}(\Xi))$$

for some valuation V^b_{Q}. In this case, $\underline{m}^b = 0$. As V^b_{Q} plays a different role than a valuation for quasilinear utility functions, we call V^b_{Q} a *quasivaluation* in the context of quasilogarithmic utility functions.

In Example 3, buyers would never choose to hit $m = \underline{m}^b$. To see this, note that as money amount m approaches $\underline{m}^b$, the buyer's utility approaches $-\infty$. Such preferences capture the possibility that agents find it increasingly difficult to borrow near a borrowing constraint.

2.3 Demand and substitutability

We next define agents' Marshallian and Hicksian demand correspondences in our context and introduce the two restrictions on preferences, the second of which we use for our existence result.

For notational convenience in this section, let us first define a *spending indicator* for agent i by

$$\chi^i = \begin{cases} 1 & \text{if } i \in B \\ -1 & \text{if } i \in S \end{cases}.$$

An *income* for an agent i is a constant w such that if i is a buyer, then $w > \underline{m}^i$. While income of sellers can be arbitrary, we ensure that buyers always start with some disposable income.

We now define *Marshallian demand*, which gives the sets of trades that maximize the utility of an agent with income w at a price vector $\mathbf{p} \in \mathbb{R}^{\Omega_i}$. Formally, correspondence $D^i_{\mathrm{M}} : \mathbb{R}^{\Omega_i} \times \mathbb{R} \rightrightarrows \mathcal{P}(X_i)$ is given by

$$D^i_{\mathrm{M}}(\mathbf{p}, w) = \operatorname*{arg\,max}_{\Xi \subseteq \Omega_i} U^i \left(\Xi, w - \chi^i \sum_{\xi \in \Xi} p_\xi \right).$$

To understand the role of the parameter w, note that w does not affect Marshallian demand when utility is quasilinear, but does for preferences such as quasilinear utility with a hard budget constraint and quasilogarithmic utility.

We now recall the standard notion of gross substitutability (see, e.g., Hatfield et al. (2013) and Fleiner et al. (2019)).

Definition 1. We say that U^i is *gross substitutable at income* w if for all trades $\omega \neq \psi \in \Omega_i$, price vectors $\mathbf{p}$, and price increments $\lambda > 0$ with $D^i_{\mathrm{M}}(\mathbf{p}, w) = \{\Xi\}$ and $D^i_{\mathrm{M}}(\mathbf{p} + \chi^i \lambda \mathbf{e}^\omega, w) = \{\Xi'\}$, if $\psi \in \Xi$, then $\psi \in \Xi'$.[18]

Intuitively, gross substitutability requires that for buyers (resp. sellers), if the price of a trade ω increases (resp. decreases), then all other trades ψ become weakly more desirable. To understand why gross substitutability is particularly restrictive outside the context of quasilinear utility, we revisit an example of quasilinear utility with a hard budget constraint that we discussed in the Introduction.

Example 4 (Failure of gross substitutability due to a hard budget constraint). Consider a buyer b for whom $\Omega_b = \{\zeta, \psi\}$, and suppose that b has quasilinear utility with a hard budget constraint (as in Example 2) with $V^b(\Xi) = 5|\Xi|$. Consider the price vectors $\mathbf{p}$ and $\mathbf{p}'$ defined by $p_\psi = 1$, $p'_\psi = 2$, and $p_\zeta = p'_\zeta = 3$. With a income of $w = 4$, we have that $D^b_{\mathrm{M}}(\mathbf{p}, w) = \{\{\zeta, \psi\}\}$ but that $D^b_{\mathrm{M}}(\mathbf{p}', w) = \{\{\psi\}\}$. Thus, raising the price of ψ can make b stop demanding ζ. This gross complementarity arises due to an income effect: when b is demanding ψ, raising the price of ψ lowers her disposable income—which can affect her willingness to pay for ζ. This complementarity occurs despite the additivity of valuation V^b.

We now consider a different notion of substitutability that is better adapted to settings outside the standard context of quasilinear utility. To do so, we first define *Hicksian demand*, which gives the sets of trades that minimize an agent's expenditure required to achieve utility of at least u at a price vector $\mathbf{p}$. Formally, the Hicksian

[18] Here, $\mathbf{e}^\omega$ denotes the elementary basis vector defined by $\mathbf{e}^\omega = (1_\omega, 0_{\Xi \smallsetminus \{\omega\}})$.

demand correspondence $D^i_{\mathrm{H}} : \mathbb{R}^{\Omega_i} \times \mathbb{R} \rightrightarrows \mathcal{P}(X_i)$ is given by

$$D^i_{\mathrm{H}}(\mathbf{p}; u) = \left\{ \Xi^* \middle| (\Xi^*, m^*) \in \operatorname*{arg\,min}_{(\Xi,m) | U^i(\Xi,m) \geq u} \left\{ m + \chi^i \sum_{\xi \in \Xi} p_\xi \right\} \right\}.$$

Hicksian demand captures agents' substitution effects. As in classical consumer theory, there is a duality between Marshallian and Hicksian demands in our setting (see Lemma B.1 in Appendix B). The combination of indivisible trades and hard budget constraints means that Marshallian demand can be discontinuous, and can be a proper subset of Hicksian demand even at positive prices—contrasting the classical setting with divisible goods where Marshallian demand is continuous and coincides with Hicksian demands for positive prices.

To see why Marshallian and Hicksian demands may not coincide in our setting, consider a buyer b for a trade ω who has quasilinear utility with a hard budget constraint, values the trade at \$2 and has an income of \$1. At the price of \$1, the buyer's Marshallian demand is $\{\{\omega\}\}$ (which delivers a utility level of 2). But at the price of \$1 and a utility level of 1, the buyer's Hicksian demand is $\{\varnothing, \{\omega\}\}$; these bundles deliver utility levels of 1 and 2 respectively. Once the price increases above \$1, Marshallian demand changes discontinuously to $\{\varnothing\}$, while Hicksian demand changes upper hemicontinuously to $\{\varnothing\}$.

The possible discontinuity of Marshallian demand renders the direct analysis of utility-maximizing choices problematic in our setting, and therefore taking the Hicksian perspective is essential. Indeed, hard budget constraints are the main cause of discontinuity of Marshallian demand; as we show in Appendix A, in cases where agents' incomes are sufficient for the budget constraint not to bind, Marshallian and Hicksian demands do coincide and are upper hemicontinuous.[19]

We can now introduce *net substitutability*—our main assumption on preferences. Net substitutability says that, for all utility levels and starting at prices where Hicksian demand is single-valued, if the price of a trade increases (resp. decreases), then the seller's (resp. buyer's) Hicksian demand for other trades decreases. That is, we require that agents view the trades as substitutable after they have been compensated for the price change (i.e., restored to their original utility level). Formally, the

[19]Moreover, when utility functions are quasilinear, Marshallian and Hicksian demands not only coincide but do not even depend on the wealth or utility levels (hence, we can refer to them simply as "demand").

definition of net substitutability is analogous to the definition of gross substitutability but places conditions on Hicksian demand for a fixed utility level rather than on Marshallian demand for a fixed income.

Definition 2. We say that U^i is *net substitutable* if for all utility levels u, trades $\omega \neq \psi \in \Omega_i$, price vectors $\mathbf{p}$, and price increments $\lambda > 0$ with $D^i_{\mathrm{H}}(\mathbf{p}; u) = \{\Xi\}$ and $D^i_{\mathrm{H}}(\mathbf{p} + \chi^i \lambda \mathbf{e}^\omega; u) = \{\Xi'\}$, if $\psi \in \Xi$, then $\psi \in \Xi'$.

When agents' utilities are quasilinear, net substitutability is equivalent to gross substitutability since the Marshallian and Hicksian demand correspondences coincide. As a result we can simply refer to agents having *substitutable* valuations, and the "net" and "gross" qualifiers can be dropped.

To illustrate the distinction between gross substitutability and net substitutability in the presence of budget constraints, let us return to Example 4. In that example, consider the utility level obtained by the buyer spending all of her income on both trades

$$u = \max_{\Xi \subseteq \Omega_b} U^b \left(\Xi, w - \sum_{\xi \in \Xi} p_\xi \right) = U^b\left(\{\zeta, \psi\}, 0\right) = 10.$$

Observe that following the price change from $\mathbf{p} = (1, 3)$ to $\mathbf{p}' = (2, 3)$, we have that $D^b_{\mathrm{H}}(\mathbf{p}; u) = D^b_{\mathrm{H}}(\mathbf{p}'; u)$—so the violation of gross substitutability does not give rise to a violation of net substitutability. This is because net substitutability places conditions on compensated price changes, so changing the price of one trade does not affect the affordability of another trade. More generally, the utility function in Example 4 is net substitutable (see Appendix D for a formal proof).

Intuitively, gross substitutabilty places conditions on agents' substitution and income effects while net substitutability only places conditions on substitution effects. In fact, subject to a mild regularity condition for buyers, the gross substitutability condition implies the net substitutability condition.

Proposition 1. *Suppose that i is a seller, or that i is a buyer and U^i is strictly increasing in trades away from utility level $-\infty$. If U^i is gross substitutable at all incomes, then U^i is net substitutable.*

Thus, with income effects or budget constraints, net substitutability is a weaker condition than gross substitutability. Indeed, gross substitutability is highly restrictive when agents demand multiple goods and have budget constraints (see Example 4). By contrast, net substitutability allows for both gross substitutability and gross complementarity between trades.[20]

Baldwin et al. (2020) proved a version of Proposition 1 without our regularity assumption or the monotonicity Assumption 4, but did not allow for hard budget constraints, and required gross substitutability even for money endowments $w \leq \underline{m}^i$.[21] The main difficulty in the proof Proposition 1 is to overcome the possible non-coincidence of Marshallian and Hicksian demands for buyers when hard budget constraints might bind.

3 Nonexistence of competitive equilibrium

We next turn to the possibility of existence of competitive equilibrium in our model. To state the definition of competitive equilibrium, we first define an *arrangement* to consist of a set Ξ of trades and a vector $\mathbf{p}$ of prices.

Definition 3. Given an income profile $(w^i)_{i \in I}$, an arrangement $[\Xi; \mathbf{p}]$ is a *competitive equilibrium* if $\Xi_i \in D^i_{\mathrm{M}}(\mathbf{p}^i, w^i)$ for all agents i.

Our definition of competitive equilibrium is standard. A competitive equilibrium consists of a set of executed trades and a price for every trade such that the market for each trade clears—i.e., that each trade is either demanded by both of its counterparties or neither.

We next give two examples showing how the combination of indivisibility of trades and the presence of hard budget constraints can lead to the nonexistence of competitive equilibrium. This nonexistence arises due to the discontinuity of buyers' Marshallian demands as their budgets are exhausted.

The first example is well-known (see, e.g., Herings and Zhou (2019)).

Example 5 (Competitive equilibria may not exist even with unit demand). As depicted in Figure 1, there is one seller s and two buyers b_1, b_2. Each buyer b_j can interact with s via a unique trade ω_j. Each buyer has quasilinear utility with a hard budget constraint with $V^{b_j}(\{\omega_j\}) = 2$. The seller has a quasilinear utility with $V^s(\varnothing) = V^s(\{\omega_j\}) = 0$ and $V^s(\{\omega_1, \omega_2\}) = -\infty$, so can execute at most one trade.

[20] In fact, net substitutability allows, for example, a buyer views a pair of trades as gross complements while the seller views the same pair of trades as gross substitutes.

[21] The latter hypothesis is implicit in Baldwin et al.'s (2020) proof of their Proposition 1.

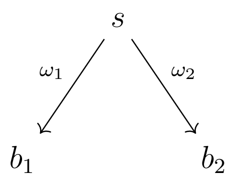

Figure 1: Trades in Example 5.

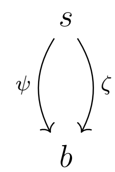

Figure 2: Trades in Examples 6 and 9.

When each buyer has an income of \$1, there is no competitive equilibrium. Indeed, note that in competitive equilibrium, the prices of the trades must be equal (to, say, p), but if $p > 1$ the trades are under-demanded and if $p \leq 1$ the trades are over-demanded.

Example 5 is knife-edge. If we perturb buyer b_1's income to $\$1+\epsilon$, then there is are competitive equilibria in which both trades have a price of, say, $\$1 + \frac{\epsilon}{2}$, and one of the trades is executed. Nevertheless, the following example shows that the non-existence of competitive equilibrium with hard budget constraints persists for generic budgets.

Example 6 (Competitive equilibria may not exist even for generic budgets). As depicted in Figure 2, there is one seller s and one buyer b, who interact via two trades ψ, ζ. The buyer's utility function is

$$U^b(\Xi, m) = \begin{cases} m & \text{if } m \geq 0 \text{ and } |\Xi| = 0 \\ m + \min\{m, 1\} & \text{if } m \geq 0 \text{ and } |\Xi| = 1 \\ m + 1 + \min\{m, 1\} & \text{if } m \geq 0 \text{ and } |\Xi| = 2 \\ -\infty & \text{if } m < 0 \end{cases}.$$

The seller s has quasilinear utility with $V^s(\Xi) = 0$ for all $\Xi \subseteq \Omega$.

When the buyer has an income $0 \leq w^b < 1$, there is no competitive equilibrium. To see why, note that the Pareto efficiency of competitive equilibria entails that exactly one trade must be realized in competitive equilibrium. Without loss of generality, suppose that ψ is realized. For b (resp. s) to demand ψ, we must have that $p_\psi \leq p_\zeta$ (resp. $p_\zeta \leq p_\psi$); it follows that $p_\psi = p_\zeta$ in equilibrium. But if $p_\psi = p_\zeta \leq \frac{w^b}{2}$, then b demands both trades; if $p_\psi = p_\zeta > \frac{w^b}{2}$, then b demands neither trade.

4 Existence of stable outcomes

The possibility that competitive equilibrium may not exist in our model motivates us to consider alternative solution concepts. In this section, we show that *stable outcomes*—a standard solution concept from matching theory—always exist as long as prices are flexible.

4.1 Stable outcomes

Rather than taking prices as given and unilaterally selecting utility-maximizing bundles, we now instead assume that agents can contract on trades and prices. Formally, a *contract* is a pair (ω, p) of a trade ω and a price p for ω (Hatfield et al., 2013). For a set of contracts $Y \subseteq X$, we let $\tau(Y) = \{\omega \in \Omega \mid (\omega, p) \in Y \text{ for some } p\}$ denote the set of trades that are associated with contracts in Y. Given a set $Y \subseteq X$ of contracts and an agent $i \in I$, let Y_i denote the set of trades in Y in which i is involved (as either buyer or seller).

An *outcome* is a set $Y \subseteq X$ of contracts such that each trade is associated with at most one price in Y—formally, $|\tau(Y)| = |Y|$. Unlike for competitive equilibrium, outcomes do not specify prices of unrealized trades.

To define stability, we need two further pieces of notation. Given an agent i, an income w, and an outcome $Z \subseteq X_i$, we let

$$\mathsf{U}^i(Z, w) = U^i\left(\tau(Z), w - \chi^i \sum_{(\omega,p)\in Z} p\right)$$

denote the utility that i achieves from the set Z of contracts given income w. We can then define the choice correspondence $C^i : \mathcal{P}(X_i) \rightrightarrows \mathcal{P}(X_i)$ by

$$C^i(Y, w) = \underset{\text{outcomes } Z \subseteq Y}{\arg\max}\ \mathsf{U}^i(Z, w);$$

here $C^i(Y, w)$ consists of agent i's most-preferred sets of contracts from Y.

We can now recall the definition of stability.

Definition 4 (Roth, 1984; Hatfield and Milgrom, 2005; Hatfield et al., 2013)**.** Given an income profile $(w^i)_{i\in I}$:

- An outcome A is *individually rational* if $A_i \in C^i(A_i, w^i)$ for all agents i.

- A nonempty set $Z \subseteq X \smallsetminus A$ *blocks* an outcome A if for all agents i and all choices $Y \in C^i(A_i \cup Z_i, w^i)$, we have that $Z_i \subseteq Y$.

- An outcome is *stable* if it is individually rational and there is no blocking set.

An outcome is stable if all agents choose all their contracts in the outcome given their incomes, and there is no blocking set of other contracts that all agents would choose when given access to their existing and blocking contracts.[22]

The following theorem is the main result of this paper.

Theorem 1. *Under net substitutability, for all income profiles, stable outcomes exist.*

We discuss the proof of Theorem 1 below, but we first return to two examples of non-existence of competitive equilibrium from the previous section to illustrate how considering stable outcomes restores existence. In Example 5, there are two stable outcomes $\{(\omega_1, 1)\}$ and $\{(\omega_2, 1)\}$, in which one trade is executed at a price of \$1. Here, the first outcome cannot be supported in competitive equilibrium as buyer b_2 would like to buy ω_2 for \$1 and the seller cannot sell to both buyers; there is no block because b_2 offering \$1 to the seller would not make her strictly prefer to sell to b_2 rather than b_1. Similarly, in Example 6, there are also two stable outcomes $\{(\psi, \frac{w^b}{2})\}$ and $\{(\zeta, \frac{w^b}{2})\}$, in which one trade is executed at a price of $\frac{w^b}{2}$.

It is worth emphasizing how general Theorem 1 is from the point of view of agents' preferences. Hatfield and Kojima (2008) showed that gross substitutability between contracts with different counterparties is necessary (in a maximal domain sense) for the existence of stable outcomes *even in many-to-one* markets without flexible prices. As net substitutability permits gross complementarities across contracts even with different counterparties, net substitutable preferences do not generally satisfy Hatfield and Kojima's (2008) necessary condition.

4.2 Relationship to previous existence results

In models of two-sided matching under gross substitutability, the existence of stable outcomes does not depend on whether prices of trades are flexible. In particular, under gross substitutability, even if the price of each trade were fixed in advance, stable outcomes would exist (Roth, 1984; Hatfield and Milgrom, 2005). If prices were

[22] Here, we agents can retain, or unilaterally drop, some or any of their existing contracts.

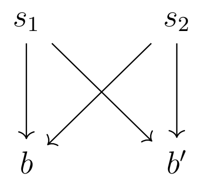

Figure 3: Trades in Example 7. We denote the trade between s_k and b (resp. b') by ω_k (resp. ω'_k).

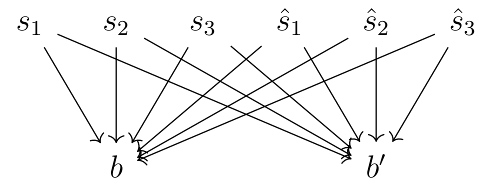

Figure 4: Trades in Example 8. We denote the trade between s_j and b (resp. b') by ω_j (resp. ω'_j), and the trade between $\hat{s}_j$ and b (resp. b') by $\hat{\omega}_j$ (resp. $\hat{\omega}'_j$).

made more flexible in such a matching market, the efficiency of the outcome could improve—this point is at the heart of Crawford's (2008) proposal to introduce flexible salaries into the National Resident Matching Program.

However, under net substitutability, price flexibility is crucial even for the existence, rather than just for the efficiency, of stable outcomes. Indeed, the following example shows that if we made prices rigid in our model, then stable outcomes could cease to exist.

Example 7. As depicted in Figure 3, there are two sellers s_1 and s_2 and two buyers b and b' who interact via four trades. Intuitively, the market is a many-to-one matching market in which the buyers are firms and the sellers are workers who are only interested in working for at most one firm. Buyer b has a quasilogarithmic utility function (as in Example 3) with an endowment of $w^b = 10$. Buyer b' has a quasilinear utility function. The quasivaluation of b and the valuation of b' are given by[23]

$$\begin{aligned} V^b_{\mathrm{Q}}(\varnothing) &= -10 & V^{b'}(\varnothing) &= 0 \\ V^b_{\mathrm{Q}}(\{\omega_1\}) &= -4 & V^{b'}(\{\omega'_1\}) &= 6 \\ V^b_{\mathrm{Q}}(\{\omega_2\}) &= -7 & V^{b'}(\{\omega'_2\}) &= 7 \\ V^b_{\mathrm{Q}}(\{\omega_1,\omega_2\}) &= -1 & V^{b'}(\{\omega'_1,\omega'_2\}) &= 7. \end{aligned}$$

The sellers have quasilinear utility functions with valuations such that they can each only participate in one trade, and s_1 (resp. s_2) has a reservation value of \$1 for ω_1

[23] Both V^b_{Q} and $V^{b'}$ are substitutable as valuations. Buyer b's utility function is therefore net substitutable as the quasilogarithmic utility functions with substitutable quasivaluations are net substitutable (see Example 7 in Baldwin et al. (2020)).

(resp. ω_2') and \$0 for ω_2 (resp. ω_2').[24]

Suppose that prices were rigid and the prices of all trades were set at \$4. In this case, it turns out that there are no stable outcomes. Indeed, considering the contracts $x_k = (\omega_k, 4)$ and $x_k' = (\omega_k', 4)$ for $k \in \{1, 2\}$, the buyers' preferences over bundles of contracts are

$$\begin{aligned} b: &\qquad \{x_1, x_2\} \succ_b \{x_1\} \succ_b \varnothing \succ_b \{x_2\} \\ b': &\qquad \{x_2'\} \succ_{b'} \{x_1'\} \succ_{b'} \varnothing \succ_{b'} \{x_1', x_2'\}, \end{aligned}$$

while the sellers' preferences over bundles of contracts are

$$\begin{aligned} s_1: &\qquad \{x_1'\} \succ_{s_1} \{x_1\} \succ_{s_1} \varnothing \succ_{s_1} \{x_1, x_1'\} \\ s_2: &\qquad \{x_2\} \succ_{s_2} \{x_2'\} \succ_{s_2} \varnothing \succ_{s_2} \{x_2, x_2'\}. \end{aligned}$$

With these preferences, there is no stable outcome among the contracts x_1, x_1', x_2, x_2'.[25] This non-existence arises due to the gross complementarity between x_1 and x_2 for b.

By contrast, Theorem 1 guarantees that there is a stable outcome when prices are flexible. For example, the outcome $\{(\omega_1, 5), x_2'\}$ is stable. Intuitively, raising the price of ω_1 mitigates the gross complementarity between ω_1 and ω_2—leading to existence.

Inspired by the seminal work of Gale and Shapley (1962), in all existing models of matching markets with a finite number of agents, the existence of stable outcomes is found by a constructive argument.[26] Under gross substitutability, a stable matching can be found by the Deferred Acceptance algorithm (Kelso and Crawford, 1982;

[24] Formally, the sellers' valuations are given by

$$\begin{aligned} V^{s_1}(\varnothing) &= 0 & V^{s_2}(\varnothing) &= 0 \\ V^{s_1}(\{\omega_1\}) &= -1 & V^{s_2}(\{\omega_2\}) &= 0 \\ V^{s_1}(\{\omega_1'\}) &= 0 & V^{s_2}(\{\omega_2'\}) &= -1 \\ V^{s_1}(\{\omega_1, \omega_1'\}) &= -\infty & V^{s_2}(\{\omega_2, \omega_2'\}) &= -\infty. \end{aligned}$$

[25] See, e.g., Alva (2013, Chapter 2). Indeed, note that any outcome involving both x_1 and x_1', or both x_2 and x_2', is not individually rational (for s_1 and s_2, respectively). And $\{x_2\}$ and $\{x_1', x_2\}$ is not individually rational (for b), while $\{x_1\}$ and $\{x_1, x_2'\}$ are blocked by $\{x_2\}$, and $\{x_1, x_2\}$ is blocked by $\{x_1'\}$.

[26] Topological arguments are sometimes used to prove existence results in models with a continuum of agents (Azevedo and Hatfield, 2018; Che et al., 2019; Greinecker and Kah, 2021; Jagadeesan and Vocke, 2021).

Roth, 1984); under weaker substitutability conditions in models of matching with contracts, the Cumulative Offer process of Hatfield and Milgrom (2005) can work where Deferred Acceptance might fail (e.g., in the case of "bilateral" substitutes introduced by Hatfield and Kojima (2010)). When prices are explicit in the matching model, these algorithms operate similarly to a monotone (i.e., ascending or a descending) auction. The following example shows that the Deferred Acceptance algorithm and the Cumulative Offer process do not generally work under net substitutability.

Example 8. As depicted in Figure 4, there are two buyers b and b' and six sellers $s_1, s_2, s_3, \hat{s}_1, \hat{s}_2, \hat{s}_3$; each buyer and seller can interact via a unique trade. Intuitively, the market is a many-to-one matching market in which there are two identical firms b, b', and three copies of each of two types of workers (s_1, s_2, s_3 are of one type, and $\hat{s}_1, \hat{s}_2, \hat{s}_3$ are of the other). Each seller's utility function is quasilinear, with valuations such that they can each only participate in one trade; sellers s_1, s_2, s_3 have reservation values of \$2, and sellers $\hat{s}_1, \hat{s}_2, \hat{s}_3$ have reservation values of \$1.[27] Each buyer has a quasilogarithmic utility function (as in Example 3) with an endowment of $w^b = w^{b'} = 10$. Buyer b's quasivaluation is given by

$$V^b_{\mathrm{Q}}(\Xi) = f(|\Xi \cap \{\omega_1, \omega_2, \omega_3\}|) + g(|\Xi \cap \{\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3\}|) - 10,$$

where

$$\begin{aligned} f(0) &= 0 & \qquad g(0) &= 0 \\ f(1) &= 4 & g(1) &= 2 \\ f(2) = f(3) &= 6 & g(2) = g(3) &= 3. \end{aligned}$$

Intuitively, workers of each type are identical, and the quasivaluation is additive across types and concave within type.[28] Buyer b''s quasivaluation is analogous.

[27] Formally, sellers' valuations are given by

$$\begin{aligned} V^{s_j}(\varnothing) &= 0 & \qquad V^{\hat{s}_j}(\varnothing) &= 0 \\ V^{s_j}(\{\omega_j\}) &= -2 & V^{\hat{s}_j}(\{\hat{\omega}_j\}) &= -1 \\ V^{s_j}\left(\{\omega'_j\}\right) &= -2 & V^{\hat{s}_j}(\{\hat{\omega}'_k\}) &= -1 \\ V^{s_j}\left(\{\omega_j, \omega'_j\}\right) &= -\infty & V^{\hat{s}_j}\left(\{\hat{\omega}_j, \hat{\omega}'_j\}\right) &= -\infty. \end{aligned}$$

[28] The quasivaluation V^b_{Q} is substitutable as a valuation. Buyer b's utility function is therefore

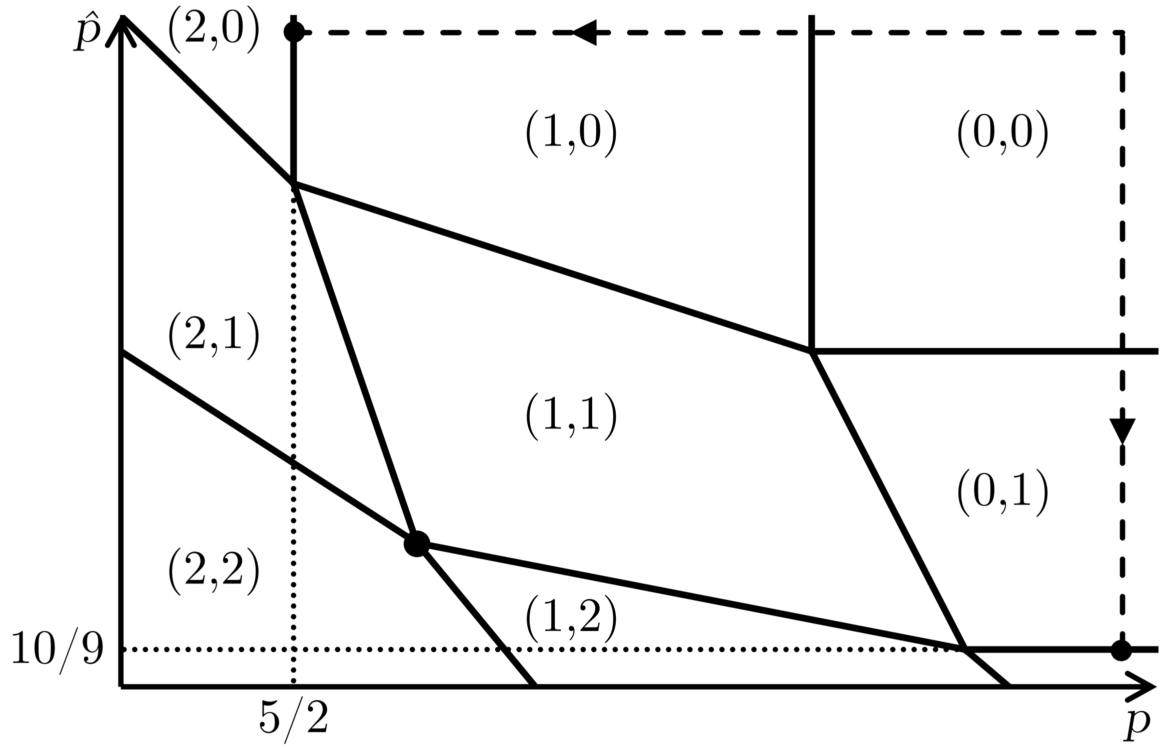

Figure 5: Stable outcomes and trajectories of matching processes in Example 8. The figure depicts the buyers' demands as a function of p and $\hat{p}$, where p denotes the prices of trades involving sellers s_1, s_2, s_3 (which are assumed to be equal for the purposes of this figure), and $\hat{p}$ denotes the prices of trades involving sellers $\hat{s}_1, \hat{s}_2, \hat{s}_3$ (which are likewise assumed to be equal). The intercepts of the axes are set at the reservation values $p = 2$ and $\hat{p} = 1$; the figure is otherwise drawn to scale. The solid black lines partition the price space into the region where trades are uniquely demanded by each buyer, and are labeled by $(x, \hat{x})$, where x (resp. $\hat{x}$) denotes the number of trades with s_1, s_2, s_3 (resp. $\hat{s}_1, \hat{s}_2, \hat{s}_3$) that are demanded. The prices in the unique stable outcome are represented by a black dot at $(p, \hat{p}) = (2\frac{6}{7}, 1\frac{3}{7})$. The dashed lines represent two possible trajectories of the Deferred Acceptance algorithm, the descending salary adjustment process, and the Cumulative Offer process discussed in Example 8. If s_1, s_2, s_3 made their offers first, then the prices of those offers would decrease along the horizontal dashed arrow until $p = \frac{5}{2}$, but then offers would have to be retracted to reach a stable outcome, as p would have fallen too far. If $\hat{s}_1, \hat{s}_2, \hat{s}_3$ made their offers first, then the prices of those offers would decrease along the vertical dashed arrow until $\hat{p} = \frac{10}{9}$, but then offers would have to be retracted to reach a stable outcome, as $\hat{p}$ would have fallen too far.

There is an essentially unique stable outcome in this example. More precisely, the stable outcomes are the ones in which each seller matches with three workers, two of one type and one of the other, sellers s_1, s_2, s_3 are paid $\$2\frac{6}{7}$, and sellers $\hat{s}_1, \hat{s}_2, \hat{s}_3$ are paid $\$1\frac{3}{7}$.[29]

However, the Deferred Acceptance (DA) algorithm (Gale and Shapley, 1962), the descending salary adjustment (DSA) process (Kelso and Crawford, 1982), and the Cumulative Offer (CO) process (Hatfield and Milgrom, 2005) may not find stable outcomes. Indeed, suppose that sellers s_1, s_2, s_3 start by making offers in DA or CO until they no longer wish to make further offers, or, equivalently, that their prices would be decreased first under DSA until all of them are matched or their reservation values reached. Then, those sellers would have to offer to match at a price of $2\frac{1}{2}$ in DA or CO, or their prices would have to be decreased to that level in DSA, before all of them would be matched. But as $2\frac{1}{2} < 2\frac{6}{7}$, offers would have to be retracted under DA or CO, or prices would need to rise under DSA, to find a stable matching—which the processes do not allow. A similar conclusion would apply if $\hat{s}_1, \hat{s}_2, \hat{s}_3$ instead made offers first under DA or CO, or their prices would be decreased first under DSA.[30] Figure 5 depicts buyers' demand and the trajectories of these processes.[31]

net substitutable as the quasilogarithmic utility functions with substitutable quasivaluations are net substitutable (see Example 7 in Baldwin et al. (2020)).

[29]These outcomes correspond to competitive equilibria in which s_1, s_2, s_3 are paid $2\frac{6}{7}$, and $\hat{s}_1, \hat{s}_2, \hat{s}_3$ are paid $1\frac{3}{7}$—which Figure 5 depicts. (As we show in Appendix A, stable outcomes generally correspond to competitive equilibria when lower bounds on money consumption cannot be hit.)

[30]In this case, they would have to make offers to match at a price of $1\frac{1}{9}$ in DA or CO, or their prices have to be decreased to that level in DSA, before all of them would be matched. But as $1\frac{1}{9} < 1\frac{3}{7}$, offers would have to be retracted under DA or CO, or prices would need to rise under DSA, to find a stable matching—which the processes do not allow.

[31]In this particular example, Sun and Yang's (2009) double-track adjustment procedure would find a stable outcome as the sellers can be partitioned into two groups within which they are gross substitutes and between which they are gross complements (Sun and Yang, 2006). These two groups are $\{s_1, s_2, s_3\}$ and $\{\hat{s}_1, \hat{s}_2, \hat{s}_3\}$. The double-track adjustment procedure would operate by starting the salaries of s_1, s_2, s_3 at a high level, and the salaries of $\hat{s}_1, \hat{s}_2, \hat{s}_3$ at a low level, and then decreasing the former and increasing the latter. However, this approach would also fail in a suitable extension with three types of workers instead of two, as Sun and Yang's (2009) approach relies on the partitioning goods into exactly two groups within which goods are gross substitutes and between which goods are gross complements (Sun and Yang, 2006). We focus on a two-type example for sake of simplicity, and to enable a graphical depiction of the trajectories of standard matching processes.

4.3 Quasiequilibrium and the proof of Theorem 1

Having emphasized the role of flexible prices in our model and shown the inadequacy of standard methods for proving the existence of a stable outcome, we now discuss the proof of Theorem 1.

The proof involves analyzing a solution concept called "quasiequilibrium" (Debreu, 1962) from the general equilibrium theory literature.

Definition 5. An arrangement $[\Xi; \mathbf{p}]$ is a *quasiequilibrium* if for each agent i, writing

$$u^i = U^i\left(\Xi, \chi^i \sum_{\xi \in \Xi} p_\xi\right),$$

we have that $u^i > -\infty$ and that $\Xi_i \in D^i_{\mathrm{H}}(\mathbf{p}; u^i)$.

In a quasiequilibrium, all agents choose their expenditure-minimizing bundles and, as in a competitive equilibrium, all markets clear. For instance, in Example 5, arrangements $[\{\omega_1\}; (1,1)]$ and $[\{\omega_2\}; (1,1)]$ are both quasiequilibria; while in Example 6, arrangements $[\{\psi\}; (\frac{w^b}{2}, \frac{w^b}{2})]$ and $[\{\zeta\}; (\frac{w^b}{2}, \frac{w^b}{2})]$ are both quasiequilibria. The correspondence of these quasiequilibria to stable outcomes turns out not to be a coincidence in these examples, but is rather a general feature of our model.

The first step of the proof of Theorem 1 is to show that quasiequilibria exist under net substitutability.

Proposition 2. *Under net substitutability, for all income profiles, quasiequilibria exist.*

To prove Proposition 2, we adapt and combine Hatfield et al.'s (2013) arguments to show the existence of competitive equilibrium in trading networks with transferable utility with a topological fixed-point argument of Baldwin et al. (2020). Specifically, we first modify sellers' utility functions to bound prices from above (Hatfield et al., 2013). We then apply a topological fixed-point argument to solve for a profile of quasiequilibrium utility levels: we adjust utility levels of agents to make agents' (quasi-)equilibrium expenditures equal their money endowments.[32]

[32] Baldwin et al. (2020) used a similar argument to show the existence of competitive equilibrium in settings with income effects but without hard budget constraints.

The second step of the proof of Theorem 1 involves showing that each outcome associated with a quasiequilibrium is stable. Formally, given an arrangement $[\Xi; \mathbf{p}]$, we define an associated outcome by

$$\kappa([\Xi; \mathbf{p}]) = \{(\xi, p_\xi) \mid \xi \in \Xi\}.$$

That is, $\kappa([\Xi; \mathbf{p}])$ is the outcome at which the trades in Ξ are realized at the prices given by $\mathbf{p}$. An outcome A is a *quasiequilibrium outcome* if $A = \kappa([\Xi; \mathbf{p}])$ for some quasiequilibrium $[\Xi; \mathbf{p}]$. A quasiequilibrium outcome only retains contracts for all the realized trades at their equilibrium prices (i.e., it does not include any contracts for unrealized trades or contracts for trades at any price other than quasiequilibrium prices).

The following proposition serves as the second step of the proof of Theorem 1.

Proposition 3. *For all income profiles, every quasiequilibrium outcome is stable.*

The proof of Proposition 3 adapts arguments of Hatfield et al. (2013) and Fleiner et al. (2019) demonstrating analogous results for competitive equilibrium; the key difference is that our argument relies on the monotonicity Assumption 4.

Theorem 1 follows by simply combining Proposition 2 with Proposition 3.

5 Other cooperative solution concepts

We next turn to the relationships between stability and two classic solution concepts: core and pairwise stability. Flexible prices will play a crucial role in both of these relationships.

5.1 The core

The first classic cooperative solution concept we consider is the core.

Definition 6. An outcome A is *core unblocked* if there do not exist a non-empty set $J \subseteq I$ of agents and a set $Z \subseteq X$ of contracts such that $\mathsf{b}(Z) \cup \mathsf{s}(Z) \subseteq J$ and $\mathsf{U}^i(Z_i, w^i) > \mathsf{U}^i(A_i, w^i)$ for all $i \in J$. An outcome is in the *core* if it is core unblocked.

An outcome is in the core if there is no set of contracts that strictly improves the utility of all agents involved in these contracts. In core blocks, unlike for blocks in the sense of Definition 4, agents may not retain any of their existing contracts with agents outside the blocking coalition. And core outcomes are weakly Pareto-efficient by construction.

The following result can be viewed as a version of the First Fundamental Theorem of Welfare Economics for our setting.

Theorem 2. *Under net substitutability, for all income profiles, every stable outcome is in the core.*

The conclusion of Theorem 2 relies crucially on price flexibility.[33] Indeed, it is well known that in many-to-many matching markets with rigid prices stable outcomes may be outside the core (Blair, 1988). Hence, Theorem 2 shows that price flexibility can improve the efficiency of stable outcomes even in the presence of budget constraints and gross complementarities. As a result, stability can be used a solution concept in market settings with flexible prices, such as auctions, in which efficiency is important.

To prove Theorem 2, we establish a partial converse to Proposition 3.

Proposition 4. *Under net substitutability, for all income profiles, every stable outcome is a quasiequilibrium outcome.*

Unlike Proposition 3, Proposition 4 relies on net substitutability, and its proof is based on applying Hatfield et al.'s (2013, 2021) and Fleiner et al.'s (2019) analogous results in a suitably defined auxiliary transferable utility economy. To complete the proof of Theorem 2, we show that quasiequilibrium outcomes are in the core—a conclusion that again relies on the monotonicity Assumption 4.[34]

5.2 Pairwise stability

Stability requires that agents have a lot of ability of coordinate on their blocking sets. If agents were less able to coordinate, it might be reasonable to only require that outcomes be immune to pairwise blocks.

[33] Assuming that prices are flexible, Hatfield et al. (2013) and Fleiner et al. (2019) showed a version of Theorem 2 under gross substitutability.

[34] The argument actually shows that all quasiequilibrium outcomes (and hence, under net substitutability, all stable outcomes) are in fact *strongly group stable* (in the sense of Hatfield et al. (2013)).

Definition 7 (Gale and Shapley, 1962)**.** Given a profile of incomes, an outcome is *pairwise stable* if it is individually rational and there is no blocking set that consists of a single contract.

By definition, every stable outcome is pairwise stable. Under gross substitutability, in many-to-many matching markets with either rigid or flexible prices, stable outcomes coincide with pairwise stable outcomes (Hatfield and Kominers, 2017; Hatfield et al., 2021; Fleiner et al., 2019). The following theorem shows that the same relationship holds under net substitutability in our model.

Theorem 3. *Under net substitutability, for all income profiles, every pairwise stable outcome is stable.*

This result may appear surprising at first: it shows that agents can focus simply on pairwise deviations to block an unstable outcome (such as an outcome that is not weakly Pareto-efficient) even though net substitutability allows for gross complementarities. However, the sufficiency of pairwise deviations to block an unstable outcome in the presence of gross complementarities in our model relies crucially on price flexibility. Under gross substitutability, each contract in a blocking set constitutes a single-contract pairwise block. By contrast, in our model, the single-contract block might need to be executed at a price that is different from the one specified in the blocking set—as the following example shows.

Example 9 (A blocking set that does not contain a blocking contract under net substitutability)*.* As depicted in Figure 2, there is one seller s and one buyer b, who interact via two trades, ζ and ψ. Define valuations

$$\begin{aligned} V^b(\varnothing) &= 0 & \qquad V^s(\varnothing) &= 0 \\ V^b(\{\zeta\}) &= 20 & V^s(\{\zeta\}) &= -2 \\ V^b(\{\psi\}) &= 1 & V^s(\{\psi\}) &= -2 \\ V^b(\{\psi, \zeta\}) &= 21 & V^s(\{\psi, \zeta\}) &= -4. \end{aligned}$$

Letting $\tilde{U}^b(\Xi, m) = V^b(\Xi) + m$ and $\tilde{U}^s(\Xi, m) = V^s(\Xi) + m$, we define

$$U^s(\Xi, m) = \tilde{U}^s(\Xi, m) + \begin{cases} 0 & \text{if } \zeta \notin \Xi \text{ or } \tilde{U}^s(\Xi, m) \geq 11 \\ 10(\tilde{U}^s(\Xi, m) - 11) & \text{otherwise} \end{cases}$$

$$U^b(\Xi, m) = \tilde{U}^b(\Xi, m) + \begin{cases} 0 & \text{if } \psi \notin \Xi \text{ or } \tilde{U}^b(\Xi, m) \leq 0 \\ 10\tilde{U}^b(\Xi, m) & \text{otherwise} \end{cases}$$

Intuitively, these utility functions are defined from the quasilinear utility functions by introducing income effects at low (resp. high) utility levels for b (resp. s) with respect to the realization of ψ (resp. ζ). In Appendix D, we show that these utility functions are indeed net substitutable.

Suppose that both b and s have incomes of 0. In this case, the autarky outcome is blocked by $\{(\zeta, 10), (\psi, 10)\}$. But neither $(\zeta, 10)$ nor $(\psi, 10)$ is a block on its own. Indeed, agents' preferences over bundles of contracts in $\{(\zeta, 10), (\psi, 10)\}$ are given by

$$\begin{aligned} b: &\quad \{(\zeta, 10), (\psi, 10)\} \succ_b \{(\zeta, 10)\} \succ_b \varnothing \succ_b \{(\psi, 10)\} \\ s: &\quad \{(\zeta, 10), (\psi, 10)\} \succ_s \{(\psi, 10)\} \succ_s \varnothing \succ_s \{(\zeta, 10)\}, \end{aligned}$$

so $(\zeta, 10)$ (resp. $(\psi, 10)$) would not be desirable to s (resp. b) on its own.

In particular, if prices were fixed at 10, then the autarky outcome would be pairwise stable but unstable. However, with flexible prices, there exist blocking contracts—such as (ζ, p) for $12 < p < 20$.

The following proposition generalizes the key takeaways from Example 9.

Proposition 5. *Under net substitutability, for all income profiles, if a set Z of contracts blocks an individually rational outcome A, then there exists a contract (ω, p_ω) blocking A for which $\omega \in \tau(Z)$.*

Thus, price flexibility is both necessary and sufficient (under net substitutability) for agents to be able to focus on single-contract blocks and still ensure stability.

6 Properties of the set of stable outcomes

We finally turn to the analysis of the structure of stable outcomes under net substitutability. When preferences satisfy gross substitutability, stable outcomes are known

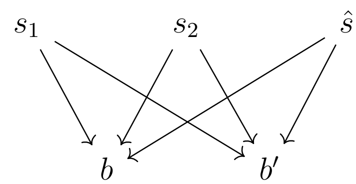

Figure 6: Trades in Example 10. We denote the trade between s_j and b (resp. b') by ω_j (resp. ω'_j), and the trade between $\hat{s}$ and b (resp. b') by $\hat{\omega}$ (resp. $\hat{\omega}'$).

to have a striking structure. First, stable outcomes form a lattice (Fleiner, 2003; Hatfield and Kominers, 2017).[35] Second, under the additional assumption of the "Law of Aggregate Demand" (Hatfield and Milgrom, 2005),[36] if an agent receives strictly more than her autarky payoff in one stable outcome, then she must participate in every stable outcome (Jagadeesan et al., 2020).[37] Third, when agents on one side of the market each demand at most one trade, stable outcomes can be implemented by a mechanism that is strategy-proof for all agents on that side (Hatfield and Milgrom, 2005; Hatfield and Kominers, 2012, 2017). The most general versions of these results for settings with flexible prices are due to Schlegel (2021).

The following example shows that all of these properties can fail under net substitutability, even though stable outcomes always exist.

Example 10. As depicted in Figure 6, there are two buyers b, b' and three sellers $s_1, s_2, \hat{s}$, and each buyer and seller can interact via a unique trade. Intuitively, the market is a many-to-one matching market in which two sellers s_1, s_2 are identical. Each seller's utility function is quasilinear, with valuations such that they can each only participate in one trade, and such that s_1, s_2 have reservation values of \$0 and $\hat{s}$

[35]In particular, there exist buyer-optimal and seller-optimal outcomes.

[36]In our context, the "Law of Aggregate Demand" would require that as prices of trades rise, each buyer (resp. seller) must demand fewer (resp. more) trades.

[37]This result relies on the absence of budget constraints (Herings and Zhou, 2019). A more general version of this "Lone Wolf Theorem", called the "Rural Hospital Theorem," states without flexible prices, agents execute the same number of trades in all stable outcomes (Hatfield and Kominers, 2012, 2017).

has a reservation value of \$4.[38] Buyer b has a quasilogarithmic utility function with quasivaluation given by

$$V^b_{\mathrm{Q}}(\Xi) = \begin{cases} -10 & \text{if } \Xi = \varnothing \\ -4 & \text{if } \Xi \cap \{\omega_1, \omega_2\} \neq \varnothing \text{ and } \hat{\omega} \notin \Xi \\ -7 & \text{if } \Xi \cap \{\omega_1, \omega_2\} = \varnothing \text{ and } \hat{\omega} \in \Xi \\ -1 & \text{if } \Xi \cap \{\omega_1, \omega_2\} \neq \varnothing \text{ and } \hat{\omega} \in \Xi \end{cases},$$

and an endowment of $w^b = 10$.[39] Intuitively, the quasivaluation V^b_{Q} values autarky at -10, and places marginal values of 6 on executing at least one of ω_1, ω_2, and 3 on ω'. Buyer b' has a quasilinear utility function with a (substitutable) valuation

$$V^{b'}(\Xi) = \begin{cases} 0 & \text{if } \Xi = \varnothing \\ 6 & \text{if } \Xi \cap \{\omega'_1, \omega'_2\} \neq \varnothing \\ 3 & \text{if } \Xi \cap \{\omega'_1, \omega'_2\} = \varnothing \text{ and } \hat{\omega}' \in \Xi \end{cases}.$$

To understand the failure of lattice structure, we calculate the stable outcomes that are most preferred by each seller. Note that s' would never trade for less than \$4, and neither s_1 nor s_2 would trade for a negative price. In this case, neither buyer would engage in trade with either s_1 or s_2 for a price of greater than \$6. There are exactly two stable outcomes in which s_1 or s_2 is paid \$6, namely $A = \{(\omega_1, 6), (\omega'_2, 6)\}$ and $A' = \{(\omega'_1, 6), (\omega'_2, 6)\}$. These outcomes are strictly preferred by s_1 and s_2 to all other stable outcomes. On the other hand, there are also stable outcomes in which $\hat{s}$ is matched at above her reservation value, such as $\hat{A} = \{(\omega_1, 0), (\omega'_2, 0), (\hat{\omega}, 7\frac{1}{2})\}$. Therefore, there exists no stable outcome that is unanimously preferred by all sellers

[38] Formally, the sellers' valuations are given by

$$\begin{aligned} V^{s_j}(\varnothing) &= 0 & V^{\hat{s}_1}(\varnothing) &= 0 \\ V^{s_j}(\{\omega_j\}) &= 0 & V^{\hat{s}_j}(\{\hat{\omega}\}) &= -4 \\ V^{s_j}\left(\{\omega'_j\}\right) &= 0 & V^{\hat{s}_j}(\{\hat{\omega}'\}) &= -4 \\ V^{s_j}\left(\{\omega_j, \omega'_j\}\right) &= -\infty & V^{\hat{s}}(\{\hat{\omega}, \hat{\omega}'\}) &= -\infty. \end{aligned}$$

[39] The quasivaluation V^b_{Q} is substitutable as a valuation. Buyer b's utility function is therefore net substitutable as the quasilogarithmic utility functions with substitutable quasivaluations are net substitutable (see Example 7 in Baldwin et al. (2020)).

to all other stable outcomes (as a result the lattice structure of stable outcomes also fails). Moreover, seller $\hat{s}$ gets more than her autarky payoff in outcome $\hat{A}$, but does not trade in outcome A.[40]

Finally, we show that there is no mechanism that implements stable outcomes that is strategy-proof for sellers. The details of the argument are in Appendix D. To see why, note that if any stable outcome other than A or A' were selected, s_1 or s_2 could profitably misreport a reservation value of $\$6-\varepsilon$. Indeed, in the market defined by such a misreport by either seller, both s_1 and s_2 would be matched and paid at least $\$6-\varepsilon$ each. On the other hand, if A or A' were selected, then $\hat{s}$ could profit by misreporting that her reservation value for $\hat{\omega}'$ were \$5 and that her reservation value for $\hat{\omega}$ were $\$\varepsilon$. Indeed, in the market defined by such a report, $\hat{s}$ would be matched with b at a salary of at least \$5 in every stable outcome. Intuitively, misreporting the reservation value for $\hat{\omega}'$ lowers the price that s_1 and s_2 can demand from b', which in turn lowers s_1 and s_2's payments from b, and thereby raises $\hat{s}$'s payment from b due to a gross complementarity.

7 Conclusion

Competitive prices may fail to clear markets with indivisibilities and budget constraints. We showed that, under the net substitutability condition, markets with indivisibilities and budget constraints nevertheless admit stable outcomes. Price flexibility plays a crucial role in our existence and efficiency results, and in the ability of agents to focus on simple blocking contracts. Net substitutability allows agents to view trades as either gross complements or gross substitutes—thereby substantially weakening known conditions for the existence of stable outcomes in finite markets. However, the structural properties of stable outcomes under gross substitutability can fail under net substitutability.

[40]Hence, the conclusion of the "Lone Wolf Theorem" fails. Unlike in Herings and Zhou (2019), this failure is not driven by the possibility of agents hitting their lower bounds on money consumption.

And indeed, the "law of aggregate demand" also holds in this example: increasing the price of a trade can never increase the total number of demanded trades. To see why, consider buyer b. Her quasivaluation implies that she views ω_1 and $\hat{\omega}$ as gross complements; ω_2 and $\hat{\omega}$ as gross complements; and ω_1 and ω_2 as perfect substitutes. If the price of $\hat{\omega}$ increased, b would never start demanding either ω_1 or ω_2; if the price of ω_1 (resp. ω_2) increased, b might switch from demanding ω_1 (resp. ω_2)to ω_2 (resp. ω_1), but would never start demanding $\hat{\omega}$. Hence, the failure of the conclusion of the Lone Wolf Theorem is driven entirely by gross complementarities between $\hat{\omega}$ and the other trades.

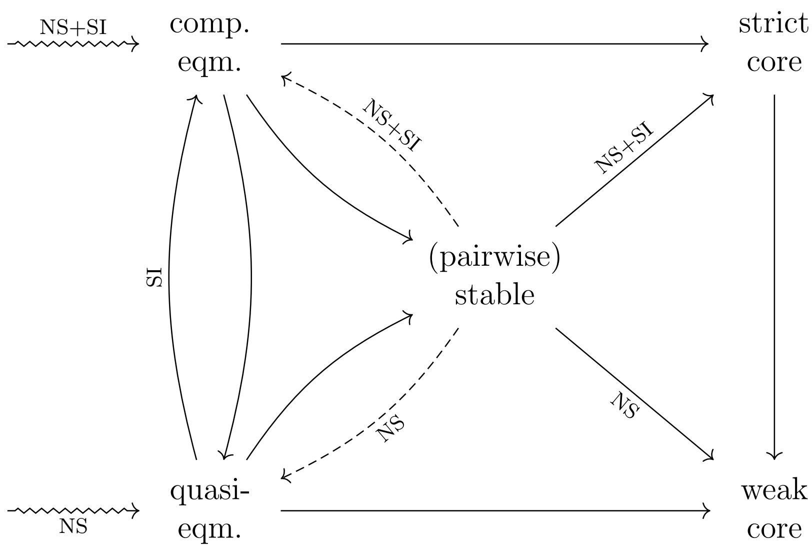

Figure 7: Summary of our results. Squiggly arrows represent existence results, and solid arrows represent relationships between solution concepts. Arrow are labeled by hypothesis, abbrevated as NS = net substitutability; SI = sufficient incomes (defined in Appendix A). Implicit in the figure is the coincidence of stability and pairwise stability under net substitutability (Theorem 3).

In Appendix A, we explore a special case of our model on which buyers' budget constraints never bind (e.g., as in Example 3). In that case, competitive equilibrium outcomes exist, are in the strict core, and coincide with stable outcomes under net substitutability. Figure 7 summarizes our results.

Our results have interesting implications for the design of auctions with budget constraints. We showed that in the presence of budget constraints, dynamic auctions may not find desirable outcomes. However, our existence and efficiency results suggest that, by using stability as a solution concept, there is scope for adapting sealed-bid auction designs (e.g., Klemperer (2010) and Milgrom (2009)) to settings with budget constraints—despite the resulting gross complementarities.

In future theoretical work, one could also put additional restrictions on net substitutability to recover some of the structure of classic matching markets; revisit our results in a model of trading networks or with transaction frictions (as in Fleiner et al. (2019) and Schlegel (2021)); explore algorithms for finding stable outcomes under net substitutability; and consider whether net substitutability forms a maximal domain of preferences for the existence of stable outcomes.

A The case with sufficient incomes

In this appendix, we present stronger results for a special case in which buyers' incomes are large enough that their budget constraints do not bind. Formally, we say that an income w for a buyer b is *sufficient* if $U^b(\varnothing, w) > U^b\left(\Xi, \underline{m}^b\right)$. A sufficient income profile is an income profile $(w^i)_{i \in I}$ such that for each buyer b, we have that w^b is a sufficient income for b.

The key point underlying this special case is that competitive equilibrium and quasiequilibrium coincide for sufficient incomes.

Lemma A.1 (Competitive equilibrium versus quasiequilibrium)**.**

(a) For all income profiles, every competitive equilibrium is a quasiequilibrium.

(b) For all sufficient income profiles, every quasiequilibrium is a competitive equilibrium.

In particular, under the sufficient incomes condition, Proposition 2 specializes to an existence result for competitive equilibrium.

Corollary A.1. *Under net substitutability, for all sufficient income profiles, competitive equilibria exist.*

Under sufficiency, Proposition 3 and Lemma A.1 give a connection between stability and competitive equilibrium. To state the connection, analogously to quasiequilibrium outcomes, we call an outcome A a *competitive equilibrium outcome* if $A = \kappa([\Xi; \mathbf{p}])$ for some competitive equilibrium $[\Xi; \mathbf{p}]$.

Corollary A.2. *Under net substitutability, for all sufficient income profiles, an outcome is stable if and only if it is a competitive equilibrium outcome.*

For our two-sided matching market setting, Corollary A.2 generalizes analogous results for substitutes valuations (Theorem 5 in Hatfield et al. (2013)), and for gross substitutable utility functions (Theorem E.1 in Fleiner et al. (2019)).

With sufficient incomes, we can also show a stronger version of the First Fundamental Theorem of Welfare Economics. Let us first strengthen the definition of the core by requiring that only one agent in the block needs to be made strictly better off (the other agents can remain indifferent).

Definition A.1. An outcome A is *strict-core unblocked* if there do not exist a nonempty set $J \subseteq I$ of agents and a set $Z \subseteq X \smallsetminus A$ of contracts such that $\mathsf{b}(Z) \cup \mathsf{s}(Z) \subseteq J$ and $\mathsf{U}^i(Z_i, w^i) \geq \mathsf{U}^i(A_i, w^i)$ for all $i \in J$ and $\mathsf{U}^i(Z_i, w^i) > \mathsf{U}^i(A_i, w^i)$ for some $i \in J$. An outcome is in the *strict core* if it is strict-core unblocked.

Outcomes in the strict core are strictly Pareto-efficient. The definition of the strict core features prominently in the standard analysis of competitive equilibrium outcomes with divisible goods. The proof of the following fact carries over to the indivisible good case.

Fact A.1 (see, e.g., Proposition 18.B.1 in Mas-Colell et al., 1995)**.** *For all income profiles, every competitive equilibrium outcome is in the strict core.*

By combining Corollary A.2 and Fact A.1, we obtain that a somewhat stronger efficiency result than Theorem 2 for the case with sufficient incomes.

Corollary A.3. *Under net substitutability, for all sufficient income profiles, every stable outcome is in the strict core.*

B Understanding Hicksian demand

We first develop a new relationship between Marshallian and Hicksian demand in our setting with indivisible goods and hard budget constraints.

Lemma B.1. *Let i be an agent. Let $\mathbf{p}$ be a price vector.*

(a) For all incomes w, we have that $D^i_{\mathrm{M}}(\mathbf{p}, w) = D^i_{\mathrm{H}}(\mathbf{p}; u)$, where

$$u = \max_{\Xi \subseteq \Omega_i} U^i \left(m - \chi^i \sum_{\xi \in \Xi} p_\xi, . \right)$$

(b) For all utility levels u, writing

$$w = \min_{(\Xi, m) | U^i(\Xi, m) \geq u} \left\{ m + \chi^i \sum_{\xi \in \Xi} p_\xi \right\}$$

if w is an income for i, then we have that $D^i_{\mathrm{H}}(\mathbf{p}; u) \supseteq D^i_{\mathrm{M}}(\mathbf{p}, w)$. If furthermore i is a seller, or i is a buyer and w is a sufficient income for i, then we have that $D^i_{\mathrm{H}}(\mathbf{p}; u) = D^i_{\mathrm{M}}(\mathbf{p}, w)$.

We next extend the quasilinear interpretation of Hicksian demand developed by Baldwin et al. (2020) to settings with hard budget constraints.[41] Formally, given an agent i and a utility level u, we define a valuation $V^i_{\mathrm{H}}(\cdot; u)$ by

$$V^i_{\mathrm{H}}(\Xi; u) = \max_{U^i(\Xi, m) \geq u} \{-m\},$$

which we call agent i's *Hicksian valuation at utility level* u. Assumptions 3, 4, and 5 ensure that $V^i_{\mathrm{H}}(\Xi; u)$ is in fact a valuation and that it varies continuously with u (when the range $\mathbb{R} \cup \{-\infty\}$ of Hicksian valuations is equipped with the topology inherited from the topology of the extended real line).[42] The following result relates the Hicksian valuations to Hicksian demand.

Lemma B.2 (Lemma 1 in Baldwin et al. (2020)[43])**.** *Let i be an agent. For all price vectors $\mathbf{p}$ and utility levels u, we have that*

$$D^i_{\mathrm{H}}(\mathbf{p}; u) = \operatorname*{arg\,max}_{\Xi \subseteq \Omega_i} \left\{ V^i_{\mathrm{H}}(\Xi; u) - \chi^i \sum_{\xi \in \Xi} p_\xi \right\}.$$

Net substitutability can be expressed as a condition on the Hicksian valuations. Specifically, U^i is net substitutable if and only if each of agent i's Hicksian valuations is substitutable (in the sense of Hatfield et al. (2013, 2019); see Remark 1 in Baldwin et al. (2020)).

We also consider Baldwin et al.'s (2020) Hicksian economies. Formally, the *Hicksian economy for a profile $(u^i)_{i \in I}$ of utility levels* is the economy in which each agent i has quasilinear utility without a budget constraint and $V^i = V^i_{\mathrm{H}}(\cdot; u)$. In light of Lemma B.2, agents' demands in each Hicksian economy are given by their Hicksian demands in the original economy, as in Baldwin et al. (2020). We use the construction of the Hicksian economy in several proofs.

[41] Baldwin et al. (2020) implicitly assume that all incomes are sufficient (see Appendix A), as they in effect assume that $U^i\left(\Xi, \underline{m}^i\right) = -\infty$ always holds.

[42] Here, Assumption 5 ensures that sellers' Hicksian valuations never take value ∞, and that buyers' Hicksian valuations never evaluate to $-\infty$ at $\varnothing$.

[43] While Baldwin et al. (2020) considered an exchange economy and imposed additional conditions on utility functions, an identical argument carries over to our setting.

C Proofs

C.1 Proof of Proposition 1

Our proof follows the structure of the proof of Proposition 1 in Baldwin et al. (2020), but allows for the possibility of hard budget constraints for buyers. The proof uses the following fact regarding valuations. Here, we write D^i for the demand correspondence for a valuation V^i.

Fact C.1. *A valuation V^i is substitutable if and only if for all price vectors $\hat{\mathbf{p}}$ with $|D^i(\hat{\mathbf{p}})| = 2$, writing $D^i(\hat{\mathbf{p}}) = \{\Xi, \Xi'\}$, we have that $|\Xi \smallsetminus \Xi'|, |\Xi' \smallsetminus \Xi| < 1$.*

For the main argument, we actually prove the contrapositive of the proposition. Suppose that U^i is not net substitutable; we show that there must exist an income for which U^i is not gross substitutable. We consider the cases of sellers and buyers separately.

Case of sellers. Suppose that i is a seller. By construction, there exists a utility level u such that $V^i_{\mathrm{H}}(\cdot; u)$ is not a substitutable valuation. Hence, by Lemma B.2 and the "if" direction of Fact C.1 for $V^i = V^i_{\mathrm{H}}(\cdot; u)$, there exists a price vector $\hat{\mathbf{p}}$ such that $|D^j_{\mathrm{H}}(\hat{\mathbf{p}}; u)| = 2$, and writing $D^j_{\mathrm{H}}(\hat{\mathbf{p}}; u) = \{\Xi, \Xi'\}$, we have that $|\Xi \smallsetminus \Xi'| \geq 2$ or that $|\Xi' \smallsetminus \Xi| \geq 2$. Without loss of generality, we can assume that $|\Xi \smallsetminus \Xi'| \geq 2$. Suppose that $\omega \in \Xi \smallsetminus \Xi'$ and that $\psi \in \Xi \smallsetminus \Xi' \smallsetminus \{\omega\}$.

Consider the income

$$w = -\sum_{\xi \in \Xi} \hat{p}_\xi - V^i_{\mathrm{H}}(\Xi; u) = -\sum_{\xi \in \Xi'} \hat{p}_\xi - V^i_{\mathrm{H}}(\Xi'; u);$$

Lemma B.1(b) implies that $D^j_{\mathrm{M}}(\hat{\mathbf{p}}, w) = \{\Xi, \Xi'\}$. Let μ be such that

$$D^i_{\mathrm{M}}(\hat{\mathbf{p}} - \mu \mathbf{e}^\omega, w), D^i_{\mathrm{M}}(\hat{\mathbf{p}} + \mu \mathbf{e}^\omega, w) \subseteq \{\Xi, \Xi'\};$$

such a μ exists due to the upper hemicontinuity of D^j_{M} (which in turn follows from Assumptions 1 and 3).

Let $\mathbf{p} = \hat{\mathbf{p}} + \mu \mathbf{e}^\omega$, let $\lambda = 2\mu$, and let $\mathbf{p}' = \mathbf{p} - \lambda \mathbf{e}^\omega = \hat{\mathbf{p}} - \mu \mathbf{e}^\omega$. We now show that

$D^i_{\mathrm{M}}(\mathbf{p}, w) = \{\Xi\}$ and that $D^i_{\mathrm{M}}(\mathbf{p}', w) = \{\Xi'\}$. We have that

$$\begin{aligned} U^i\left(\Xi, w + \sum_{\xi \in \Xi} p_\xi\right) &> U^i\left(\Xi, w + \sum_{\xi \in \Xi} \hat{p}_\xi\right) \\ &= U^i\left(\Xi', w + \sum_{\xi \in \Xi'} \hat{p}_\xi\right) \\ &= U^i\left(\Xi', w + \sum_{\xi \in \Xi'} p_\xi\right), \end{aligned}$$

where the inequality holds due to Assumption 4 because $p_\omega > \hat{p}_\omega$ and $\omega \in \Xi$, the first equality holds because $\{\Xi, \Xi'\} \subseteq D^j_{\mathrm{M}}(\hat{\mathbf{p}}, w)$, and the second equality holds because $\omega \notin \Xi'$. Similarly, we have that

$$\begin{aligned} U^i\left(\Xi, w + \sum_{\xi \in \Xi} p'_\xi\right) &< U^i\left(\Xi, w + \sum_{\xi \in \Xi} \hat{p}_\xi\right) \\ &= U^i\left(\Xi', w + \sum_{\xi \in \Xi'} \hat{p}_\xi\right) \\ &= U^i\left(\Xi', w + \sum_{\xi \in \Xi'} p'_\xi\right), \end{aligned}$$

where the inequality holds due to Assumption 4 because $p'_\omega < \hat{p}_\omega$ and $\omega \in \Xi$, the first equality holds because $\{\Xi, \Xi'\} \subseteq D^j_{\mathrm{M}}(\hat{\mathbf{p}}, w)$, and the second equality holds because $\omega \notin \Xi'$.

As $D^i_{\mathrm{M}}(\mathbf{p}, w), D^i_{\mathrm{M}}(\mathbf{p}', w) \subseteq \{\Xi, \Xi'\}$, we must have that $D^i_{\mathrm{M}}(\mathbf{p}, w) = \{\Xi\}$ and that $D^i_{\mathrm{M}}(\mathbf{p}', w) = \{\Xi'\}$. As $|\Xi \smallsetminus \Xi'| \geq 2$, there exists $\xi \in \Xi \smallsetminus \Xi' \smallsetminus \{\omega\}$, and thus U^i is not gross substitutable at income w.

Case of buyers. Now suppose that i is a buyer. We apply a similar argument to the case of sellers, but need to carefully deal with the hard budget constraints, and ensure that the income at which gross substitutability fails in fact satisfies $w > \underline{m}^i$. Our strategy is to first show that gross substitutability sharply restricts the values of utility of that lower bound $\underline{m}^i$ (Claim C.1). We then follow the approach of the case of sellers, but move prices and utility levels so the utility level is not a utility level that can be achieved with $m = \underline{m}^i$ (Claim C.2) and prices are nearly nonnegative

(Claim C.3). We then characterize how perturbing prices can affect Hicksian demand (Claim C.4), and complete the argument similarly to the case of sellers.

We first show a claim regarding the interaction between gross substitutability and utility evaluated at the lower bound $\underline{m}^i$ The claim extends the conclusion of Example 4.

Claim C.1. *Let i be a buyer, and suppose that U^i is strictly increasing in trades away from utility level $-\infty$. If U^i is gross substitutable for all incomes and $|\Omega_i| > 1$, then $U^i(\Xi, \underline{m}^i) = -\infty$ for all $\Xi \subsetneq \Omega_i$.*

Proof. Let

$$\mathcal{S} = \left\{\Xi \subseteq \Omega_i \mid U^i\left(\Xi, \underline{m}^i\right) \in \mathbb{R}\right\}$$

denote the family of sets of trades at which i can feasibly hit the lower bound $\underline{m}^i$. If $\mathcal{S} = \varnothing$, then the claim holds. Hence, we can assume that $\mathcal{S} \neq \varnothing$.

We now prove that there must exist an income at which U^i is not gross substitutable. By Assumption 4, if $\Xi \in \mathcal{S}$ and $\Psi \supseteq \Xi$, then $\Psi \in \mathcal{S}$. Hence, $\mathcal{S} \neq \{\varnothing\}$.

Let $n = \min_{\Psi \in \mathcal{S} \smallsetminus \{\varnothing\}} |\Psi|$. If $n = |\Omega_i|$, then since $|\Omega_i| > 1$, we must have that $\mathcal{S} = \{\Omega_i\}$. Now, let Ξ maximize $U^i(\Psi, \underline{m}^i)$ over all $\Psi \in \mathcal{S}$ with $|\Psi| = n$; we show that $\Xi = \Omega_i$ must hold.

Suppose for sake of deriving a contradiction that $\Xi \subsetneq \Omega_i$. Let $\psi \in \Omega_i \smallsetminus \Xi$ be arbitrary. Let w be an income such that $U^i(\Psi, w) < U^i(\Xi, \underline{m}^i)$ for all Ψ with $|\Psi| < n$, and $U^i(\Xi, w) < U^i(\Xi \cup \{\psi\}, \underline{m}^i)$; such an income exists due to the choice of Ξ, the strict monotonicity of U^i in trades away from utility level $-\infty$, and Assumption 3. Let $\omega \in \Xi$ be arbitrary, let $K = w - \underline{m}^i > 0$, and define price vectors

$$\begin{aligned}
\mathbf{p} &= \left(0_\Xi, K_\psi, (2K)_{\Omega_i \smallsetminus \Xi \smallsetminus \{\psi\}}\right) \\
\mathbf{p}' &= \mathbf{p} + \frac{K}{2}\mathbf{e}^\omega
\end{aligned}$$

We now show that $D^i_{\mathrm{M}}(\mathbf{p}, w) = \{\Xi \cup \{\psi\}\}$, let $\Psi \in D^i_{\mathrm{M}}(\mathbf{p}, w)$. By strict monotonicity in trades, we must have that $\Xi \subseteq \Psi$, and by Assumption 2, we must have that $\Psi \subseteq \Xi \cup \{\psi\}$. Since $U^i(\Xi, w) < U^i(\Xi \cup \{\psi\}, \underline{m}^i)$, it follows that $\Psi = \Xi \cup \{\psi\}$.

We next show that $D^i_{\mathrm{M}}(\mathbf{p}', w) = \{\Xi\}$, let $\Psi \in D^i_{\mathrm{M}}(\mathbf{p}', w)$. Again, by strict monotonicity in trades, we must have that $\Xi \smallsetminus \{\omega\} \subseteq \Psi$, and by Assumption 2, we must have that $\Psi \subsetneq \Xi \cup \{\psi\}$. Hence, we must have that $\Psi \in \{\Xi \smallsetminus \{\omega\}, \Xi \cup \{\psi\} \smallsetminus \{\omega\}, \Xi\}$.

But the choice of w and Assumption 4 imply that

$$U^i\left(\Xi, \underline{m}^i + \frac{K}{2}\right) > U^i\left(\Xi, \underline{m}^i\right) > U^i\left(\Xi \smallsetminus \{\omega\}, w\right),$$

and hence that $\Xi \smallsetminus \{\omega\} \notin D^i_{\mathrm{M}}(\mathbf{p}, w)$. And similarly, due to the choice of Ξ and Assumption 4, we have that

$$U^i\left(\Xi, \underline{m}^i + \frac{K}{2}\right) > U^i\left(\Xi, \underline{m}^i\right) \geq U^i\left(\Xi \cup \{\psi\} \smallsetminus \{\omega\}, \underline{m}^i\right),$$

and hence that $\Xi \cup \{\psi\} \smallsetminus \{\omega\} \notin D^i_{\mathrm{M}}(\mathbf{p}, w)$. Thus, we must have that $D^i_{\mathrm{M}}(\mathbf{p}', w) = \{\Xi\}$.

It follows that U^i is not gross substitutable at income w—a contradiction. □

The following claim provides a version of Fact C.1 that focuses on utility levels at which hard budget constraints do not bind for the bundles under consideration. Formally, let $L = \{U^i(\Psi, \underline{m}^i) \mid \Psi \subseteq \Omega_i\}$ denote the set of utility levels other than $-\infty$ that can be achieved by i by hitting her lower bound on money consumption; this set has size at most 1 by Claim C.1.

Claim C.2. *If i is a buyer and U^i is not net substitutable, then there exists a price vector $\hat{\mathbf{p}}$ and a utility level $u \notin L$ such that $|D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u)| = 2$, and writing $D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u) = \{\Xi, \Xi'\}$, we have that $|\Xi \smallsetminus \Xi'| \geq 2$.*

Proof. By construction, there exists a utility level u_0 such that $V^i_{\mathrm{H}}(\cdot; u_0)$ is not a substitutable valuation. Since the set of substitutable valuations is closed,[44] and the Hicksian valuations vary continuously with the utility level, and the set S is finite, we can assume that $u_0 \notin L$.

The "only if" direction of Fact C.1 and Lemma B.2 together imply that there is a price vector $\hat{\mathbf{p}}'$ with $|D^i_{\mathrm{H}}(\hat{\mathbf{p}}'; u_0)| = 2$ such that writing $D^i_{\mathrm{H}}(\hat{\mathbf{p}}'; u) = \{\Xi, \Xi'\}$, we have that $|\Xi \smallsetminus \Xi'| \geq 2$ or that $|\Xi' \smallsetminus \Xi| \geq 2$. By exchanging the roles of Ξ and Ξ' if necessary, we can ensure that $|\Xi \smallsetminus \Xi'| \geq 2$. □

We next show that it is sufficient to restrict consideration to price vectors that are nearly nonnegative to detect net complementarities.

[44] This property is a consequence of characterization of substitutable valuations in terms of "$M^\natural$-concavity" (see Fujishige and Yang (2003); Hatfield et al. (2019) gave a similar characterization in a matching setting).

Claim C.3. *If i is a buyer and U^i is not net substitutable, then there exist a price vector $\hat{\mathbf{p}}$ and a utility level $u \notin L$ such that $|D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u)| = 2$, and writing $D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u) = \{\Xi, \Xi'\}$, we have that $|\Xi \smallsetminus \Xi'| \geq 2$ and one of the following conditions holds:*

(1) we have that $V^i_{\mathrm{H}}(\Xi; u) = -\underline{m}^i$ and that $V^i_{\mathrm{H}}(\Xi'; u) - \sum_{\xi \in \Xi'} \min\{\hat{p}_\xi, 0\} < -\underline{m}^i$

(2) we have that $V^i_{\mathrm{H}}(\Xi; u) - \sum_{\xi \in \Xi} \min\{\hat{p}_\xi, 0\} < -\underline{m}^i$.

Proof. Claim C.2 implies that there exist a price vector $\hat{\mathbf{p}}'$ and a utility level $u \notin S$ such that $|D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u)| = 2$, and writing $D^i_{\mathrm{H}}(\hat{\mathbf{p}}'; u) = \{\Xi, \Xi'\}$, we have that $|\Xi \smallsetminus \Xi'| \geq 2$. We divide into cases to define a constant ε, which will restrict the degree of negativity of prices to be considered.

Case 1: $V^i_{\mathrm{H}}(\Xi; u) = -\underline{m}^i$. In this case, we must have that $U^i(\Xi, \underline{m}^i) \neq -\infty$. As $|\Xi \smallsetminus \Xi'| \geq 2$, we must have that $|\Omega_i| > 1$. By Claim C.1, it follows that $U^i(\Xi', \underline{m}^i) = -\infty$. Thus, we must have that $V^i_{\mathrm{H}}(\Xi'; u) = -\underline{m}^i$; let $\varepsilon > 0$ be such that $|\Omega|\varepsilon < -V^i_{\mathrm{H}}(\Xi'; u) - \underline{m}^i$.

Case 2: $V^i_{\mathrm{H}}(\Xi'; u) < -\underline{m}^i$. In this case, let $\varepsilon > 0$ be such that $|\Omega|\varepsilon < -V^i_{\mathrm{H}}(\Xi; u) - \underline{m}^i$.

Let $\Omega_\varepsilon = \{\omega \in \Omega \mid p_\omega < -\varepsilon\}$. It follows from Assumption 4 that $\Omega_\varepsilon \subseteq \Xi, \Xi'$. Define a price vector $\hat{\mathbf{p}} = (\hat{\mathbf{p}}'_{\Omega_i \smallsetminus \Omega_\varepsilon}, (-\varepsilon)_{\Omega_\varepsilon})$. Let $K = -\sum_{\omega \in \Omega_\varepsilon}(\hat{p}'_\omega + \varepsilon)$. Note that in Case 1, we have that

$$V^i_{\mathrm{H}}(\Xi'; u) - \sum_{\xi \in \Xi'} \min\{\hat{p}_\xi, 0\} \leq V^i_{\mathrm{H}}(\Xi'; u) + |\Omega|\varepsilon < -\underline{m}^i$$

and in Case 2, we have that

$$V^i_{\mathrm{H}}(\Xi; u) - \sum_{\xi \in \Xi} \min\{\hat{p}_\xi, 0\} \leq V^i_{\mathrm{H}}(\Xi; u) + |\Omega|\varepsilon < -\underline{m}^i.$$

Thus, Case (1) and (2) of the claim correspond to Cases 1 and 2 above, respectively.

It remains to show that $D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u) = \{\Xi, \Xi'\}$. Let $\Psi \in D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u)$. By Lemma B.2, we must have that

$$V^i_{\mathrm{H}}(\Psi; u) - V^i_{\mathrm{H}}(\Xi; u) \geq \sum_{\xi \in \Psi} \hat{p}_\xi - \sum_{\xi \in \Xi} \hat{p}_\xi.$$

It follows from Assumption 4 that $\Omega_\varepsilon \subseteq \Psi$. Hence, we have that

$$\sum_{\xi\in\Psi} \hat{p}_i = -K + \sum_{\xi\in\Psi} \hat{p}'_\xi.$$

But since $\Xi \supseteq \Omega_\varepsilon$, we also have that

$$\sum_{\xi\in\Xi} \hat{p}_\xi = -K + \sum_{\xi\in\Xi} \hat{p}'_\xi.$$

Hence, we must have that

$$V^i_{\mathrm{H}}(\Psi; u) - V^i_{\mathrm{H}}(\Xi; u) \geq \sum_{\xi\in\Psi} \hat{p}'_\xi - \sum_{\xi\in\Xi} \hat{p}'_\xi,$$

as since $\Xi \in D^i_{\mathrm{H}}(\hat{\mathbf{p}}'; u)$, it follows from Lemma B.2 that $\Psi \in D^i_{\mathrm{H}}(\hat{\mathbf{p}}'; u)$. Since $\Psi \in D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u)$ was arbitrary, we have shown that $D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u) \subseteq D^i_{\mathrm{H}}(\hat{\mathbf{p}}'; u) = \{\Xi, \Xi'\}$.

But since $D^i_{\mathrm{H}}(\hat{\mathbf{p}}'; u) = \{\Xi, \Xi'\}$, it follows from Lemma B.2 that

$$V^i_{\mathrm{H}}(\Xi'; u) - V^i_{\mathrm{H}}(\Xi; u) \geq \sum_{\xi\in\Xi'} \hat{p}'_\xi - \sum_{\xi\in\Xi} \hat{p}'_\xi.$$

And as $\Xi' \supseteq \Omega_\varepsilon$ as well, we have that

$$\sum_{\xi\in\Xi'} \hat{p}_\xi = -K + \sum_{\xi\in\Xi'} \hat{p}'_\xi.$$

Hence, we must have that

$$V^i_{\mathrm{H}}(\Xi'; u) - V^i_{\mathrm{H}}(\Xi; u) \geq \sum_{\xi\in\Xi'} \hat{p}_\xi - \sum_{\xi\in\Xi} \hat{p}_\xi.$$

By Lemma B.2, it follows that $D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u) = \{\Xi, \Xi'\}$. □

To complete the argument, we will also need the following claim regarding Hicksian demand.

Claim C.4. *Let i is a buyer, let $\hat{\mathbf{p}}$ be a price vector, and let u be a utility level such that $|D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u)| = 2$. Writing $D^i_{\mathrm{H}}(\hat{\mathbf{p}}; u) = \{\Xi, \Xi'\}$, let $\omega \in \Xi \smallsetminus \Xi'$. For all sufficiently small $\varepsilon > 0$, we have that $D^i_{\mathrm{H}}(\hat{\mathbf{p}} + \varepsilon \mathbf{e}^\omega; u) = \{\Xi'\}$.*

Proof. It follows from Lemma B.2 that $D^i_{\mathrm{H}}(\cdot;u)$ is upper hemicontinuous. Hence, taking ε sufficiently small and letting $\mathbf{p}' = \hat{\mathbf{p}} + \varepsilon \mathbf{e}^{\omega}$, we can ensure that

$$D^i_{\mathrm{H}}(\mathbf{p}';u) \subseteq D^i_{\mathrm{H}}(\hat{\mathbf{p}};u) = \{\Xi, \Xi'\}.$$

By Lemma B.2, we have that

$$V^i_{\mathrm{H}}(\Xi;u) - \sum_{\xi \in \Xi} \hat{p}_\xi = V^i_{\mathrm{H}}(\Xi';u) - \sum_{\xi \in \Xi'} \hat{p}_\xi.$$

Since $\omega \in \Xi \smallsetminus \Xi'$, the definition of $\mathbf{p}'$ then implies that

$$V^i_{\mathrm{H}}(\Xi;u) - \sum_{\xi \in \Xi} p'_\xi < V^i_{\mathrm{H}}(\Xi;u) - \sum_{\xi \in \Xi} \hat{p}_\xi = V^i_{\mathrm{H}}(\Xi';u) - \sum_{\xi \in \Xi'} \hat{p}_\xi = V^i_{\mathrm{H}}(\Xi';u) - \sum_{\xi \in \Xi'} p'_\xi.$$

By Lemma B.2, it follows that $\Xi \notin D^i_{\mathrm{H}}(\mathbf{p}';u)$. Hence, we must have that $D^i_{\mathrm{H}}(\mathbf{p}';u) = \{\Xi'\}$. $\square$

Now let $\hat{\mathbf{p}}, u, \Xi, \Xi'$ be as in Claim C.3. Without loss of generality, we can assume that $|\Xi \smallsetminus \Xi'| \geq 2$. Let $\omega \in \Xi \smallsetminus \Xi'$. To complete the argument, we divide into cases based on which

Case 1: $V^i_{\mathrm{H}}(\Xi;u) = -\underline{m}^i$ and $V^i_{\mathrm{H}}(\Xi;u) - \sum_{\xi \in \Xi} \min\{\hat{p}_\xi, 0\} < -\underline{m}^i$.

By construction, we have that $U^i(\Xi, \underline{m}^i) \geq u$; since $u \notin L$, we must have that $U^i(\Xi, \underline{m}^i) > u$. By contrast, since $V^i_{\mathrm{H}}(\Xi';u) < -\underline{m}^i$, it follows from Assumptions 3 and 4 that $U^i(\Xi', -V^i_{\mathrm{H}}(\Xi';u)) = u$.

Consider a scalar

$$w = \sum_{\xi \in \Xi} \hat{p}_\xi - V^i_{\mathrm{H}}(\Xi;u) = \sum_{\xi \in \Xi'} \hat{p}_\xi - V^i_{\mathrm{H}}(\Xi';u).$$

Note that

$$w \geq \sum_{\xi \in \Xi'} \min\{\hat{p}_\xi, 0\} - V^i_{\mathrm{H}}(\Xi';u) > \underline{m}^i,$$

so w is an income for i.

By Lemma B.1(b), we have that $D^i_{\mathrm{M}}(\hat{\mathbf{p}}, w) \subseteq \{\Xi, \Xi'\}$. And note that

$$U^i\left(\Xi, w - \chi^i \sum_{\xi\in\Xi} \hat{p}_\xi\right) = U^i\left(\Xi, \underline{m}^i\right) > u$$

while

$$U^i\left(\Xi', w - \chi^i \sum_{\xi\in\Xi'} \hat{p}_\xi\right) = U^i\left(\Xi', -V^i_{\mathrm{H}}(\Xi'; u)\right) = u.$$

Hence, we have that $D^i_{\mathrm{M}}(\hat{\mathbf{p}}, w) = \{\Xi\}$.

By Claim C.4, for sufficiently small $\varepsilon > 0$, letting $\mathbf{p}' = \hat{\mathbf{p}} + \chi^i \varepsilon \mathbf{e}^\omega$, we have that $D^i_{\mathrm{H}}(\mathbf{p}'; u) = \{\Xi'\}$. Since $\omega \notin \Xi'$, it follows from Lemma B.1(b) that $D^i_{\mathrm{M}}(\mathbf{p}', w) = \{\Xi'\}$. As $|\Xi \smallsetminus \Xi'| \geq 2$, there exists $\xi \in \Xi \smallsetminus \Xi' \smallsetminus \{\omega\}$, and hence U^i is not gross substitutable at income w.

Case 2: $V^i_{\mathrm{H}}(\Xi; u) - \sum_{\xi\in\Xi} \min\{\hat{p}_\xi, 0\} < -\underline{m}^i$.

Consider a scalar

$$w = \sum_{\xi\in\Xi} \hat{p}_\xi - V^i_{\mathrm{H}}(\Xi; u') = \sum_{\xi\in\Xi} \hat{p}_\xi - V^i_{\mathrm{H}}(\Xi'; u').$$

By the hypothesis of the case, we have that

$$w \geq \sum_{\xi\in\Xi} \min\{\hat{p}_\xi, 0\} - V^i_{\mathrm{H}}(\Xi; u) > \underline{m}^i.$$

Hence, w is an income for i.

By Lemma B.2, we have that

$$\{\Xi, \Xi'\} = \underset{\Psi\subseteq\Omega_i}{\arg\max}\left\{V^i_{\mathrm{H}}(\Psi; u) - \sum_{\xi\in\Psi} p_\xi\right\}. \tag{C.1}$$

Hence, there exists $\mu > 0$ such that

$$V^i_{\mathrm{H}}(\Psi; u) - \sum_{\xi\in\Psi} p_\xi < V^i_{\mathrm{H}}(\Xi; u) - \sum_{\xi\in\Xi} p_\xi - 5\mu$$

for all $\Psi \in \mathcal{P}(\Omega_i) \smallsetminus \{\Xi, \Xi'\}$.

Let $\mathbf{p} = \hat{\mathbf{p}} - \mu \mathbf{e}^{\omega}$, let $\lambda = 2\mu$, and let $\mathbf{p}' = \mathbf{p} + \lambda \mathbf{e}^{\omega} = \hat{\mathbf{p}} + \mu \mathbf{e}^{\omega}$.

In light of Claim C.4, by reducing μ if necessary, we can ensure that $D^i_{\mathrm{H}}(\mathbf{p}'; u) = \{\Xi'\}$. By Lemma B.1(b), it follows that $D^i_{\mathrm{M}}(\mathbf{p}', w) = \{\Xi'\}$.

We next show that $D^i_{\mathrm{M}}(\mathbf{p}, w) = \{\Xi\}$. Let $u' = U^i(\Xi, \mu - V^i_{\mathrm{H}}(\Xi; u)) > u$, where the inequality follows from Assumption 4. By construction, we have that $V^i_{\mathrm{H}}(\Xi; u') \geq V^i_{\mathrm{H}}(\Xi; u) - \mu$, and it follows that

$$V^i_{\mathrm{H}}(\Xi; u) - \sum_{\xi \in \Xi} \hat{p}_\xi \leq V^i_{\mathrm{H}}(\Xi; u') - \sum_{\xi \in \Xi} \hat{p}_\xi + \mu = V^i_{\mathrm{H}}(\Xi; u') - \sum_{\xi \in \Xi} p_\xi, \tag{C.2}$$

We also have that $V^i_{\mathrm{H}}(\Psi; u') \leq V^i_{\mathrm{H}}(\Psi; u)$ for all $\Psi \subseteq \Omega_i$. It follows that, for all $\Psi \in \mathcal{P}(\Omega_i) \smallsetminus \{\Xi, \Xi'\}$, we have that

$$\begin{aligned} V^i_{\mathrm{H}}(\Psi; u') - \sum_{\xi \in \Psi} p_\xi &\leq V^i_{\mathrm{H}}(\Psi; u) - \sum_{\xi \in \Psi} p_\xi \\ &\leq V^i_{\mathrm{H}}(\Psi; u) - \sum_{\xi \in \Psi} \hat{p}_\xi + \mu \\ &< V^i_{\mathrm{H}}(\Xi; u) - \sum_{\xi \in \Xi} \hat{p}_\xi \\ &\leq V^i_{\mathrm{H}}(\Xi; u') - \sum_{\xi \in \Xi} p_\xi, \end{aligned}$$

where the second inequality follows from the definition of $\mathbf{p}$, and the third inequality follows from the definition of μ, and the fourth inequality is (C.2) Hence, by Lemma B.2, we have that $D^i_{\mathrm{H}}(\mathbf{p}; u') \subseteq \{\Xi, \Xi'\}$ And since $|\Xi \smallsetminus \Xi'| \geq 2$, Claim C.1 implies that $U^i(\Xi', \underline{m}^i) = -\infty$. By Assumptions 2, 3, and 4, it follows that $V^i_{\mathrm{H}}(\Xi'; u') < V^i_{\mathrm{H}}(\Xi'; u)$. Hence, we have that

$$\begin{aligned} V^i_{\mathrm{H}}(\Xi'; u') - \sum_{\xi \in \Xi'} p_\xi &< V^i_{\mathrm{H}}(\Xi'; u) - \sum_{\xi \in \Xi'} p_\xi \\ &= V^i_{\mathrm{H}}(\Xi'; u) - \sum_{\xi \in \Xi'} \hat{p}_\xi \\ &= V^i_{\mathrm{H}}(\Xi; u) - \sum_{\xi \in \Xi} \hat{p}_\xi \\ &\leq V^i_{\mathrm{H}}(\Xi; u') - \sum_{\xi \in \Xi} p_\xi, \end{aligned}$$

where the first equality holds due to the definition of $\mathbf{p}$ since $\omega \notin \Xi'$, the second equality follows from (C.1), and the second inequality is (C.2). By Lemma B.2, we must therefore have that $D^i_{\mathrm{H}}(\mathbf{p}; u') = \{\Xi\}$. And by Lemma B.1(b), it follows that $D^i_{\mathrm{M}}(\mathbf{p}, w) = \{\Xi\}$.

As $|\Xi \smallsetminus \Xi'| \geq 2$, there exists $\xi \in \Xi \smallsetminus \Xi' \smallsetminus \{\omega\}$, and hence U^i is not gross substitutable at income w.

By construction, the cases exhaust all possibilities, and we have therefore proven that there exists an income at which U^i is not gross substitutable.

C.2 Proof of Proposition 2

To prove the proposition, we modify the market so the following technical condition is satisfied in a way that preserves net substitutability.

Assumption C.1. For each agent i, letting $\overline{u}^i = \sup_{(\Xi,m)} U^i(\Xi, m)$, we have that

$$\lim_{m\to\infty} U^i(\Xi, m) = \overline{u}^i$$

for all $\Xi \subseteq \Omega_i$.

Assumption C.1 ensures that the Hicksian valuations never take value $-\infty$, and allows us to adapt an argument of Baldwin et al. (2020) to prove the existence of quasiequilibrium under net substitutability.

Proposition C.1. *Under Assumption C.1, for all income profiles, quasiequilibria exist.*

We then show that quasiequilibria in the modified economy correspond to quasiequilibria in the original economy to complete the argument.

In the remainder of this section, we first prove Proposition 2 by exploiting Proposition C.1 in a modified market, and then prove Proposition C.1.

C.2.1 Proof of Proposition 2 assuming Proposition C.1

Consider an income profile $(w^i)_{i\in I}$. For each buyer b, define a quantity

$$M^b = \max_{\Xi\subseteq\Omega_b}\{V^b_{\mathrm{H}}\left(\Xi; U^b\left(\varnothing, w^b\right)\right) + w^b\}.$$

Assumption 4 implies that $M^b \geq 0$ for all buyers b. Let $\Pi \geq 1 + \max_{b \in B} M^b$ be an arbitrary real number.

We modify sellers' utility functions by giving them the options to dispose of any trade by paying Π. Formally, for each seller s, we define

$$\hat{U}^s(\Xi, m) = \max_{\Psi \subseteq \Xi} U^s(\Psi, m - \Pi|\Xi \smallsetminus \Psi|).$$

It is straightforward to verify that $\hat{U}^s$ satisfies Assumptions 1 holds for $\hat{U}^s$ with $\mathcal{F}^s = \mathcal{P}(\Omega_s)$ since $U^s(\varnothing, m) \in \mathbb{R}$ for all m (by Assumption 1 for U^s). Assumptions 3–5 for $\hat{U}^s$ follow from the corresponding assumptions for U^s.

Consider a modified market in which each seller s's utility function is $\hat{U}^s$, and each buyer's utility function is U^b.

Claim C.5. *Under net substitutability in the original market, net substitutability and Assumption C.1 hold in the modified market.*

Proof. Denoting the Hicksian valuation at utility level u for $\hat{U}^s$ by $\hat{V}^s_{\mathrm{H}}(\cdot; u)$, we have that

$$\hat{V}^s_{\mathrm{H}}(\Xi; u) = \max_{\Psi \subseteq \Xi} \{V^i_{\mathrm{H}}(\Psi; u) - \Pi|\Xi \smallsetminus \Psi|\} \tag{C.3}$$

for all sellers s by construction. Under net substitutability, it follows from Lemma B.2 that each Hicksian valuation $V^s_{\mathrm{H}}(\cdot; u)$ is substitutable. For each seller s, (C.3) entails that $\hat{V}^s_{\mathrm{H}}(\cdot; u)$ can be generated from $V^s_{\mathrm{H}}(\cdot; u)$ by "allowing s to produce each trade at a cost of Π" in the sense of the proof of Theorem 1 in Appendix A of Hatfield et al. (2013). Hence, Lemma A.2 in Hatfield et al. (2013) (which shows that this transformation preserves substitutability) implies that $\hat{V}^s_{\mathrm{H}}(\cdot; u)$ is substitutable for each seller s. Thus, net substitutability is satisfied in the modified market.

Next, we show that Assumption C.1 is satisfied. For buyers b, note that Assumption 5 implies

$$\lim_{m \to \infty} U^b(\varnothing, m) = \infty,$$

hence in particular that $\overline{u}^b = \infty$. But by Assumption 4, it follows that

$$\lim_{m \to \infty} U^b(\Xi, m) = \infty$$

for all $\Xi \subseteq \Omega_b$—as desired. On the other hand, for sellers s, note that

$$\hat{U}^s(\varnothing, m) \leq \hat{U}^s(\Xi, m - \Pi|\Xi|) \leq \hat{U}^s(\varnothing, m - \Pi|\Xi|)$$

for all $\Xi \subseteq \Omega_s$ by construction. By Assumption 4, the limit $\lim_{m\to\infty} U^s(\Xi, m)$ exists (in $\mathbb{R} \cup \{\infty\}$), and it follows that

$$\lim_{m\to\infty} U^s(\Xi, m) = \lim_{m\to\infty} U^s(\varnothing, m)$$

for all $\Xi \subseteq \Omega_s$. Hence, we have that

$$\lim_{m\to\infty} U^s(\Xi, m) = \overline{u}^s,$$

as desired. $\square$

By Proposition C.1, which we prove independently in the next subsection, there exists a quasiequilibrium in the modified market for the income profile $(w^i)_{i\in I}$. The following claim completes the argument.

Claim C.6. *If $[\Xi; \mathbf{p}]$ is a quasiequilibrium in the modified market for the income profile $(w^i)_{i\in I}$, then $[\Xi; \mathbf{p}]$ is a quasiequilibrium in the original market (for the same income profile).*

Proof. For each agent i, let $m^i = w^i - \chi^i \sum_{\xi\in\Xi_i} p_\xi$ and let $u^i = U^i(\Xi_i, m^i)$. By construction, for all buyers b, we have that $u^b > -\infty$ and that $\Xi_b \in D^b_{\mathrm{H}}(\mathbf{p}; u^b)$. It remains to prove that $u^s > -\infty$ and that $\Xi_s \in D^s_{\mathrm{H}}(\mathbf{p}; u^s)$ for sellers s.

For sellers s, letting

$$\hat{u}^s = \hat{U}^s\left(\Xi_s, w^s + \sum_{\xi\in\Xi_s} p_\xi\right)$$

and letting $\hat{D}^s_{\mathrm{H}}$ denote the Hicksian demand correspondence for $\hat{U}^s$, we have that $\hat{u}^s > -\infty$ and that $\Xi_s \in \hat{D}^s_{\mathrm{H}}(\mathbf{p}; u^s)$ by construction as well.

In particular, in light of Assumption 4, if $p_\xi < 0$, we would have to have that $\xi \in \Xi_{\mathsf{b}(\omega)}$ but $\xi \notin \Xi_{\mathsf{s}(\omega)}$—a contradiction. Hence, we can conclude that $p_\xi \geq 0$ must hold for all trades ξ.

We next show that $u^b \geq U^b(\varnothing, w^b)$ must hold for all buyers b. Suppose for sake of deriving a contradiction that $u^b < U^b(\varnothing, w^b)$. As w^b is an income for b, it follows from Assumption 4 that $V_{\mathrm{H}}^b\left(\varnothing; u^b\right) > -w^b$. But it also follows from Assumption 4 and the definition of u^b that

$$V_{\mathrm{H}}^b\left(\Xi_b; u^b\right) = \sum_{\xi\in\Xi_b} p_\xi - w^b.$$

Hence, we have that

$$V_{\mathrm{H}}^b\left(\Xi_b; u^b\right) - \sum_{\xi\in\Xi_b} p_\xi = -w^b < V_{\mathrm{H}}^b\left(\varnothing; u^b\right),$$

which contradicts Lemma B.1 as $\Xi_b \in D_{\mathrm{H}}^b\left(\mathbf{p}; u^b\right)$. Hence, we can conclude that $u^b \geq U^b\left(\varnothing, w^b\right)$. It follows from the definition of M^b that $m^b \geq w^b - M^b$ for all buyers b.

We now show that $p_\xi < \Pi$ must hold for all trades ξ. Suppose for sake of deriving a contradiction that $p_\omega \geq \Pi$. In light of (C.3) and Lemma B.1, we must have that $\omega \in \Xi_{\mathsf{s}(\omega)}$, and hence that $\omega \in \Xi$. But then as $p_\xi \geq 0$ for all trades ξ, we have that

$$\sum_{\xi\in\Xi_{\mathsf{b}(\omega)}} p_\xi \geq \Pi > M^{\mathsf{b}(\omega)} \geq w^{\mathsf{b}(\omega)} - m^{\mathsf{b}(\omega)} = \sum_{\xi\in\Xi_{\mathsf{b}(\omega)}} p_\xi,$$

where the strict inequality follows from the definition of Π—a contradiction. Hence, we can conclude that $p_\omega < \Pi$ must hold.

To complete the argument, let s be a seller. By Lemma B.1(b), we must have that

$$\Xi_s \in \underset{\Psi\subseteq\Omega_s}{\arg\max}\left\{\hat{U}^s\left(\Psi, w^s + \sum_{\xi\in\Psi} p_\xi\right)\right\} \tag{C.4}$$

$$= \underset{\Psi\subseteq\Omega_s}{\arg\max}\left\{\max_{\Psi'\subseteq\Psi}\left\{\hat{U}^s\left(\Psi, w^s + \sum_{\xi\in\Psi} p_\xi - \Pi|\Psi \smallsetminus \Psi'|\right)\right\}\right\} \tag{C.5}$$

$$= \left\{\Psi \;\middle|\; (\Psi, \Psi') \in \underset{\Psi'\subseteq\Psi\subseteq\Omega_s}{\arg\max}\left\{\hat{U}^s\left(\Psi, w^s + \sum_{\xi\in\Psi} p_\xi - \Pi|\Psi \smallsetminus \Psi'|\right)\right\}\right\}. \tag{C.6}$$

But since $p_\xi < \Pi$ for all trades ξ, we must have that $\Psi = \Psi'$ in every optimum in (C.6). It follows that $\Psi' = \Psi$ must hold in the inner maximization problem in (C.5) for every optimizer Ψ of the outer maximization problem. Applying this conclusion to $\Psi = \Xi$, we have that

$$-\infty \neq \hat{u}^s = \hat{U}^s\left(\Xi_s, w^s + \sum_{\xi \in \Xi_s} p_\xi\right) = U^s\left(\Xi_s, w^s + \sum_{\xi \in \Xi_s} p_\xi\right) = u^s.$$

Since $\hat{U}^s(\Xi', m') \geq U^s(\Xi', m')$ holds for all (Ξ', m') by construction, in light of (C.4), it follows that $\Xi_s \in D^s_{\mathrm{M}}(\mathbf{p}, w^s)$. Lemma B.1(a) then yields that $\Xi_s \in D^s_{\mathrm{H}}(\mathbf{p}; u^s)$—as desired. □

C.2.2 Proof of Proposition C.1

The proof follows Baldwin et al.'s (2020) argument to prove their Theorem 1, but obtains existence of quasiequilibrium instead of competitive equilibrium, and has technical some differences due to the possibility of hard budget constraints.

Consider a income profile $(w^i)_{i\in I}$. For each agent j, we define a utility level $u^i_{\min} = U^i(\varnothing, w^i)$ and let

$$K^i = w^i + \max_{\Xi \subseteq \Omega_i} V^i_{\mathrm{H}}\left(\Xi; u^i_{\min}\right),$$

which is non-negative by construction. Furthermore, let $K = 1 + \sum_{i\in I} K^i$ and let

$$u^i_{\max} = \max_{\Xi \subseteq \Omega_i} U^i\left(\Xi, w^i + K\right).$$

Given a profile $\mathbf{u} = (u^i)_{i\in I}$ of utility levels, let

$$T(\mathbf{u}) = \left\{ \left(\chi^i \sum_{\xi \in \Xi_i} p_i\right)_{i \in I} - V^i_{\mathrm{H}}\left(\Xi_i; u^i\right) - w^i \;\middle|\; \begin{array}{l} [\Xi; \mathbf{p}] \text{ is a competitive equilibrium} \\ \text{in the Hicksian economy for the} \\ \text{profile } (u^i)_{i\in I} \text{ of utility levels} \end{array} \right\}$$

denote the set of profiles of net expenditures over all competitive equilibria in the Hicksian economy for the profile $(u^i)_{i\in I}$ of utility levels.

Claim C.7 (Claim A.3 in Baldwin et al., 2020)**.** *Under Assumption C.1, there exist* $\underline{M}, \overline{M}$ *such that the correspondence* $T : \bigtimes_{i\in I}[u^i_{\min}, u^i_{\max}] \rightrightarrows \mathbb{R}^I$ *is upper hemicontinuous and has compact, convex values and range contained in* $[\underline{M}, \overline{M}]^I$*.*

Technically, Baldwin et al.'s (2020) model also assumes that $U^i(\Xi, \underline{m}^i)$ is independent of Ξ for each agent i (see Equation (1) in Baldwin et al. (2020)). However, in terms of conditions on preferences, the proof of their Claim A.3 only uses the facts that $V^i_{\mathrm{H}}(\Xi; u)$ is continuous and real-valued—properties that hold under Assumption C.1.

Claim C.8. *Under Assumption C.1 and net substitutability, there exists a profile* $\mathbf{u} = (u^i)_{i\in I}$ *of utility levels such that* $\mathbf{0} \in T(\mathbf{u})$.

The proof follows the proof of Claim A.6 in Baldwin et al. (2020), but applies in a matching market. While most of the argument is identical, there are two differences—one at the beginning of the argument and one at the end—which we highlight.

Proof. Consider the compact, convex set

$$Z = [\underline{M}, \overline{M}]^J \times \bigtimes_{j\in J} [u^j_{\min}, u^j_{\max}].$$

As $T(\mathbf{u}) \subseteq [\underline{M}, \overline{M}]^J$ for all $\mathbf{u} \in \bigtimes_{j\in J}[u^j_{\min}, u^j_{\max}]$, we can define a correspondence $\Phi : Z \rightrightarrows Z$ by

$$\Phi(\mathbf{t}, \mathbf{u}) = T(\mathbf{u}) \times \operatorname*{arg\,min}_{\hat{\mathbf{u}} \in \bigtimes_{i\in I}[u^i_{\min}, u^i_{\max}]} \left\{ \sum_{i\in I} t^i \hat{u}^i \right\}.$$

Claim C.7 guarantees that $T : \bigtimes_{i\in I}[u^i_{\min}, u^i_{\max}] \rightrightarrows \mathbb{R}^I$ is upper hemicontinuous and has compact, convex values. Since net substitutability implies that each Hicksian valuation is substitutable, Theorem 1 in Hatfield et al. (2013) guarantees that competitive equilibria exist in each Hicksian economy. Hence, T is non-empty valued.[45] Because $\bigtimes_{i\in I}[u^i_{\min}, u^i_{\max}]$ is compact and convex, it follows that the correspondence Φ is upper hemicontinuous and has non-empty, compact, convex values as well. Hence, Kakutani's Fixed Point Theorem guarantees that Φ has a fixed point $(\mathbf{t}, \mathbf{u})$.

By construction, we have that $\mathbf{t} \in T(\mathbf{u})$ and that

$$u^i \in \operatorname*{arg\,min}_{\hat{u}^i \in [u^i_{\min}, u^i_{\max}]} t^i \hat{u}^i \tag{C.7}$$

for all agents i. It suffices to prove that $\mathbf{t} = \mathbf{0}$.

Let $[\Xi; \mathbf{p}]$ be a competitive equilibrium in the Hicksian economy for the profile

[45] Here, Baldwin et al. (2020) instead directly assumed the existence of competitive equilibrium in each Hicksian economy to formulate their Theorem 1.

$(u^j)_{j \in J}$ of utility levels with

$$\chi^i \sum_{\xi \in \Xi_i} p_\xi - V^i_{\mathrm{H}}\left(\Xi_i; u^i\right) - w^i = t^i \tag{C.8}$$

for all agents i. As $u^i \geq u^i_{\min}$ and $V^i_{\mathrm{H}}(\Xi_i; \cdot)$ is weakly decreasing for each agent i (by construction), it follows from Equation (C.8) and the definition of K^i that

$$\begin{aligned} t^i &= \chi^i \sum_{\xi \in \Xi_i} p_\xi - V^i_{\mathrm{H}}\left(\Xi_i; u^i\right) - w^i \\ &\geq \chi^i \sum_{\xi \in \Xi_i} p_\xi - V^i_{\mathrm{H}}\left(\Xi_i; u^i_{\min}\right) - w^i \\ &\geq \chi^i \sum_{\xi \in \Xi_i} p_\xi - K^i \end{aligned} \tag{C.9}$$

for all agents i.

Next, we claim that $t^j \leq 0$ for all agents j. If $t^i > 0$, then Equation (C.7) would imply that $u^i = u^i_{\min}$. But as $\mathbf{t} \in T(\mathbf{u})$, it would follow that

$$t^i = \chi^i \sum_{\xi \in \Xi_i} p_\xi - V^i_{\mathrm{H}}\left(\Xi_i; u^i\right) - w^i \leq -V^i_{\mathrm{H}}\left(\varnothing; u^i_{\min}\right) - w^i = 0,$$

where the inequality holds since $[\Xi; \mathbf{p}]$ is a competitive equilibrium in the Hicksian for the profile $(u^i)_{i \in I}$ of utility levels, and the last equality holds by Assumption 4 due to the definitions of V^i_{H} and $u^i_{\min}$. Thus, we can conclude that $t^i \leq 0$ must hold for all agents j.

By construction, we have that

$$\sum_{i \in I} \left(\chi^i \sum_{\xi \in \Xi_i} p_\xi \right) = 0 \geq \sum_{i \in I} t^i,$$

where the inequality holds because $t^i \leq 0$ for all agents i. It follows that for all agents i, we have that

$$t^i - \chi^i \sum_{\xi \in \Xi_i} p_\xi \leq \sum_{j \in I \smallsetminus \{i\}} \left(\chi^j \sum_{\xi \in \Xi_j} p_\xi - t^j \right) \leq \sum_{j \in I \smallsetminus \{i\}} K^j \leq \sum_{j \in I} K^j < K,$$

where the second inequality follows from Equation (C.9), the third inequality holds because $K^j \geq 0$, and the fourth inequality holds due to the definition of K. Hence, by Equation (C.8), we have that

$$-V_{\mathrm{H}}^i\left(\Xi_i; u^i\right) = w^i + t^i - \chi^i \sum_{\xi \in \Xi_i} p_\xi < w^i + K$$

for all agents i. Since utility is strictly increasing in the consumption of money, it follows that

$$u^i \leq U^i\left(\Xi_i, -V_{\mathrm{H}}^i\left(\Xi_i; u^i\right)\right) < U^i\left(\Xi_i, w^i + K\right) \leq u_{\max}^i,$$

where the first inequality in fact holds due to the definition of V_{H}^i and the second inequality holds due to the definition of $u_{\max}^i$.[46] Equation (C.7) then implies that $t^i \geq 0$ for all agents i, so we must have that $t^i = 0$ for all agents i. □

By Claim C.8, there exists a profile $\hat{\mathbf{u}} = (\hat{u}^i)_{i \in I}$ of utility levels and a competitive equilibrium $[\Xi; \mathbf{p}]$ in the corresponding Hicksian economy with

$$w^i = -V_{\mathrm{H}}^i\left(\Xi_i; \hat{u}^i\right) + \chi^i \sum_{\xi \in \Xi_i} p_\xi \tag{C.10}$$

for all $i \in I$. Lemma B.2 implies that $\Xi_i \in D_{\mathrm{H}}^i(\mathbf{p}; \hat{u}^i)$ for all $i \in I$, and we have that $U^i\left(\Xi_i, w^i - \chi^i \sum_{\xi \in \Xi_i} p_i\right) \geq \hat{u}^i$ for all $i \in I$ by Equation (C.10).

Let $u^i = U^i\left(\Xi_i, w^i - \chi^i \sum_{\xi \in \Xi_i} p_\xi\right) \geq \hat{u}^i$. We claim that $\Xi_i \in D_{\mathrm{H}}^i(\mathbf{p}; u^i)$ must hold. Indeed, $m = w^i - \chi^i \sum_{\xi \in \Xi_i} = -V_{\mathrm{H}}^i\left(\Xi_i; \hat{u}^i\right)$, let (Ξ', m') be such that $U^i(\Xi', m') \geq u^i$. By construction, we must have that $U^i(\Xi', m') \geq \hat{u}^i$. But as $\Xi_i \in D_{\mathrm{H}}^i(\mathbf{p}; u^i)$, we must have that

$$m + \chi^i \sum_{\xi \in \Xi_i} p_\xi \leq m' + \chi^i \sum_{\xi \in \Xi'} p_\xi.$$

Since $U^i(\Xi_i, m) \geq u^i$ and (Ξ', m') was arbitrary, we can conclude that $\Xi_i \in D_{\mathrm{H}}^i(\mathbf{p}; u^i)$ must hold.

Hence, $[\Xi; \mathbf{p}]$ is a quasiequilibrium for the income profile $(w^i)_{i \in I}$.

[46] In Baldwin et al.'s (2020) context, the first inequality is in fact an equality due to the absence of hard budget constraints.

C.3 Proof of Proposition 3

We actually prove a stronger statement, which we also use in the proof of Theorem 2. To do so, we consider a strengthening of stability that is also a refinement of the core.

Definition C.1 (Hatfield et al., 2013)**.** An outcome A is *strongly unblocked* if there do not exist a non-empty set $Z \subseteq X \smallsetminus A$ and sets of contracts $Y^i \subseteq A_i \cup Z_i$ for $i \in I$ such that $Y^i \supseteq Z_i$ and $\mathsf{U}^i(Y^i, w^i) > \mathsf{U}^i(A_i, w^i)$ for all agents i with $Z_i \neq \varnothing$. An outcome is *strongly group stable* if it is individually rational and strongly unblocked.

Strongly group stable outcomes are stable and in weak core (Hatfield et al., 2013).

We next formulate a strengthening of Proposition 3 that applies to strong group stability and does not rely on net substitutability.

Proposition C.2. *For all income profiles, every quasiequilibrium outcome is strongly group stable.*

Since strongly group stable outcomes are stable, Proposition 3 follows immediately from Proposition C.2. Note that this argument shows that the "if" direction does not rely on net substitutability.

The proof of Propositions C.2 in turn builds on arguments that show that competitive equilibrium outcomes are strongly group stable (Hatfield et al., 2013; Fleiner et al., 2019); the focus on quasiequilibrium instead of competitive equilibrium, and the possibility of hard budget constraints for buyers in our model, introduce additional complexities.

Proof of Proposition C.2. We prove the contrapositive. Let $(w^i)_{i \in I}$ be an income profile, let $[\Xi; \mathbf{p}]$ be an arrangement, and suppose that $A = \kappa([\Xi; \mathbf{p}])$ is not strongly group stable. We prove that $[\Xi; \mathbf{p}]$ cannot be a quasiequilibrium.

First, suppose that A is not individually rational—i.e., that $A_i \notin C^i(A_i, w^i)$ for some agent i. For such an agent i, we must have that $\Xi_i \notin D^i_{\mathrm{M}}(\mathbf{p}, w^i)$. If i is a seller, then the contrapositive of Lemma B.1(b) implies that $\Xi_i \notin D^i_{\mathrm{H}}(\mathbf{p}; w^i)$—so $[\Xi; \mathbf{p}]$ is not a quasiequilibrium. By contrast, if i is a buyer, then if furthermore

$$\underline{m}^i + \sum_{\xi \in \Xi_i} p_\xi > w^i,$$

the arrangement $[\Xi; \mathbf{p}]$ could not be a quasiequilibrium as Assumption 2 would then entail that

$$U^i\left(\Xi_i, w^i - \sum_{\xi \in \Xi_i} p_\xi\right) = -\infty.$$

Hence, we can assume that i is a buyer and that

$$\underline{m}^i + \sum_{\xi \in \Xi_i} p_\xi \leq w^i.$$

Let $Y \in C^i(A_i, w^i)$ maximize $|W|$ over all $W \in C^i(A_i, w^i)$. By Assumption 4, we must have that $p_\xi > 0$ for each trade $\xi \in \tau(A_i \smallsetminus Y)$. It follows that

$$\underline{m}^i + \sum_{\xi \in \tau(W)} p_\xi < w^i.$$

As

$$U^i\left(\tau(W), w^i - \sum_{\xi \in \tau(W)} p_\xi\right) = \mathsf{U}^i\left(W, w^i\right) > \mathsf{U}^i\left(A_i, w^i\right) = U^i\left(\Xi_i, w^i - \sum_{\xi \in \Xi_i} p_\xi\right)$$

by construction and Assumption 3, there must exist $\epsilon > 0$ such that

$$U^i\left(\tau(W), w^i - \epsilon - \sum_{\xi \in \tau(W)} p_\xi\right) \geq U^i\left(\Xi_i, w^i - \sum_{\xi \in \Xi_i} p_\xi\right)$$

still holds. It follows that

$$\Xi_i \notin D^i_{\mathrm{H}}\left(\mathbf{p}; U^i\left(\Xi_i, w^i - \sum_{\xi \in \tau(\Xi_i)} p_\xi\right)\right),$$

so $[\Xi; \mathbf{p}]$ cannot be a quasiequilibrium.

Hence, we can assume that A is individually rational but not strongly unblocked—that is, that there exists a non-empty set of contracts $Z \subseteq X \smallsetminus A$ and, for each agent i with $Z_i \neq \varnothing$, a set of contracts $Y^i \subseteq Z_i \cup A_i$ with $Y^i \supseteq Z_i$ and $\mathsf{U}^i(Y^i, w^i) > \mathsf{U}^i(A_i, w^i)$ (see Definition C.1).

We let $J = \{i \in I \mid Z_i \neq \varnothing\}$. For each $i \in J$, we let

$$\mathcal{M}^i = \max\left\{ t \;\middle|\; U^i\left(\tau(Y^i), w^i - \chi^i \sum_{\xi \in \tau(Y^i)} p_\xi - t\right) \geq \mathsf{U}^i\left(A_i, w^i\right) \right\}$$

denote the negative of the compensating variation for i from the change from $\tau(A_i)$ to $\tau(Y^i)$ at price vector p; the maximum is defined due to Assumptions 3 and 4 and the individual rationality of A. For $\xi \in \tau(Z)$, let q_ξ be the unique price such that $(\xi, q_\xi) \in Z$. Define $q_\xi = p_\xi$ for $\xi \in \Omega \smallsetminus \tau(Z)$. For each $i \in J$, the definition of Y^i ensures that

$$U^i\left(\tau(Y^i), w^i - \chi^i \sum_{\xi \in \tau(Y^i)} q_\xi\right) > \mathsf{U}^i\left(A_i, w^i\right);$$

it follows that

$$\mathcal{M}^i - \chi^i \sum_{\xi \in \tau(Y^i)} (p_\xi - q_\xi) \geq 0.$$

Moreover, for sellers $s \in J \cap S$, it follows from Assumptions 1, 3, and 4 that

$$\mathcal{M}^s - \chi^s \sum_{\xi \in \tau(Y^s)} (p_\xi - q_\xi) > 0.$$

Because $p_\xi = q_\omega$ for $\xi \notin \tau(Z)$ and $Z_i \subseteq Y^i$ for all $i \in J$, we have that

$$\mathcal{M}^i - \chi^i \sum_{\xi \in \tau(Z_i)} (p_\xi - q_\xi) \geq 0$$

for all $i \in J$ with strict inequality for $i \in J \cap S$.

Since Z is non-empty and each trade involves a seller and a buyer, we have that $J \cap S \neq \varnothing$. Hence, summing over $i \in J$, we have that $\sum_{i \in J} \mathcal{M}^i > 0$. In particular, there exists $i \in J$ with $\mathcal{M}^i > 0$. For such i, since

$$U^i\left(\tau(Y^i), w^i - \chi^i \sum_{\xi \in \tau(Y^i)} p_\xi - \mathcal{M}^i\right) \geq \mathsf{U}^i\left(A_i, w^i\right) = U^i\left(\Xi_i, w^i - \chi^i \sum_{\xi \in \tau(\Xi_i)} p_\xi\right)$$

and U^i is strictly increasing in money away from utility level $-\infty$, we have that

$$\Xi_i \notin D^i_{\mathrm{H}}\left(\mathbf{p}; U^i\left(\Xi_i, w^i - \chi^i \sum_{\xi \in \tau(\Xi_i)} p_\xi\right)\right).$$

Therefore, $[\Xi; \mathbf{p}]$ is not a quasiequilibrium. $\square$

C.4 Proof of Proposition 4

We actually prove a stronger result that applies to pairwise stable outcomes; we also use this strengthening to prove Proposition 5.

Proposition C.3. *Under net substitutability, for all income profiles, every pairwise stable outcome is a quasiequilibrium outcome.*

Since stable outcomes are pairwise stable, Proposition 4 follows from Proposition C.3.

The key to the proof of Proposition C.3 is the following lemma.

Lemma C.1. *Let $(w^i)_{i \in I}$ be a income profile. If A is a pairwise stable outcome, then A is a pairwise stable outcome in the Hicksian economy for the profile $(u^i)_{i\in I}$ of utility levels, where $u^i = \mathsf{U}^i(A_i, w^i)$.*

Proof. Since A is individually rational, it must contain at most one contract corresponding to each trade. For each trade $\xi \in \tau(A)$, let p_ξ be such that $(\xi, p_\xi) \in A$.

We first show that A is individually rational in the Hicksian economy for the profile $(u^i)_{i\in I}$ of utility levels. Consider a modified market in which the set of all trades in $\tau(A)$ and agents' preferences are restrictions of the ones in the original market. The individual rationality of A in the original market implies the individual rationality of A in the modified market. In particular, in the modified market, we must have that $\tau(A_i) \in D^i_{\mathrm{M}}(\mathbf{p}, w^i)$ in the modified market for all agents $i \in I$. By Lemma B.1(a), we then have that $\tau(A_i) \in D^i_{\mathrm{H}}(\mathbf{p}; w^i)$ in the modified market for all agents i. In light of Lemma B.2, it follows that A is individually rational the Hicksian economy for the profile $(u^i)_{i\in I}$ of utility levels.

We next show that A is not blocked by any contract in the Hicksian economy for the profile $(u^i)_{i\in I}$ of utility levels. Let $x = (\omega, p_\omega) \in X$ be a contract with $\omega \notin \tau(A)$. Since A is pairwise stable, there exists $i \in \{\mathsf{b}(\omega), \mathsf{s}(\omega)\}$ and $Y \in C^i(A_i \cup \{x\}, w^i)$

with $x \notin Y$. Since $A_i \in C^i(A_i, w^i)$, it follows that $A_i \in C^i(A_i \cup \{x\}, w^i)$ by revealed preference. Consider a modified market in which the set of all trades in $\tau(A) \cup \{\omega\}$ and agents' preferences are restrictions of the ones in the original market. By construction, we have that $\tau(A_i) \in D^i_{\mathrm{M}}(\mathbf{p}, w^i)$ in the modified economy. By Lemma B.1(a), we then have that $\tau(A_i) \in D^i_{\mathrm{H}}(\mathbf{p}; w^i)$. It follows that x cannot block A in the Hicksian economy for the profile $(u^i)_{i\in I}$ of utility levels. Thus, A cannot be blocked by any contract $x = (\omega, p_\omega)$ with $\omega \notin \tau(A)$ in the Hicksian economy for the profile $(u^i)_{i\in I}$ of utility levels. As agents cannot choose more than one contract for any trade, A cannot be blocked by any contract in the Hicksian economy for the profile $(u^i)_{i\in I}$ of utility levels. $\square$

Proof of Proposition C.3. Fix a income profile $(w^i)_{i\in I}$ and consider a pairwise stable outcome A. Lemma C.1 guarantees that A is a pairwise stable outcome in the Hicksian economy for the profile $(u^i)_{i\in I}$ of utility levels, where $u^i = U^i(A_i, w^i)$. Theorem E.1 in Fleiner et al. (2019) (see also Corollary 1 in Appendix B of Hatfield et al. (2021)) then implies that A is a competitive equilibrium outcome in that Hicksian economy—say $A = \tau([\Xi; \mathbf{p}])$. Then, by Lemma B.2, we have that $\Xi_i \in D^i_{\mathrm{H}}(\mathbf{p}; u^i)$ for all $i \in I$—so $[\Xi; \mathbf{p}]$ is a quasiequilibrium for the income profile $(w^i)_{i\in I}$. $\square$

C.5 Proof of Theorem 2

Fix an income profile, and consider a stable outcome A. By Proposition 4, A is a quasiequilibrium outcome. By Proposition C.2, it follows that A is strongly group stable, hence in the core.

C.6 Proof of Theorem 3

Theorem 3 follows immediately from Proposition 5, which we prove independently.

C.7 Proof of Proposition 5

Let $(w^i)_{i\in I}$ be a income profile. Suppose that for that income profile, A is an individually rational outcome and a set Z of contracts blocks A. Consider a modified market in which the set of all trades is $\tau(Z \cup A)$ and agents' preferences are restrictions of the ones in the original market. In the modified market, $(w^i)_{i\in I}$ is a income profile for which A is an individually rational outcome and Z blocks A. By Proposition C.3, A

is a quasiequilibrium outcome in the modified market. Hence, by the contrapositive of Proposition C.3, A cannot be pairwise stable in the modified market. As A is individually rational (in the modified market), there must exist a contract x in the modified market that blocks A.

By construction, x must block A in the original market, and we must have that $\tau(x) \in \tau(Z \cup A)$. As agents cannot choose more than one contract with trade $\tau(x)$, we must have that $\tau(x) \notin A$. Hence, we must have that $\tau(x) \in \tau(Z)$—as desired.

C.8 Proof of Lemma A.1

Parts (a) and (b) of the lemma follow from Parts (a) and (b) of Lemma B.1, respectively.

C.9 Proof of Lemma B.1

Proof of Part (a). We first show that

$$w = \min_{(\Xi,m)|U^i(\Xi,m)\geq u} \left\{ m + \chi^i \sum_{\xi\in\Xi} p_\xi \right\} \tag{C.11}$$

and that $D^i_{\mathrm{H}}(\mathbf{p};u) \supseteq D^i_{\mathrm{M}}(\mathbf{p},w)$; the argument for this follows the proof of Claim C.1 in Baldwin et al. (2020). Letting $\Xi' \in D^i_{\mathrm{M}}(\mathbf{p},w)$ and $m' = w - \chi^i \sum_{\xi\in\Xi'} p_\xi$, we have that $U^i(\Xi',m') = u$ and that $m' + \chi^i \sum_{\xi\in\Xi'} p_\xi = w$ by construction. It follows that

$$w \geq \min_{(\Xi,m)|U^i(\Xi,m)\geq u} \left\{ m + \chi^i \sum_{\xi\in\Xi} p_\xi \right\}$$

Suppose for the sake of deriving a contradiction that there exists (Ξ'',m'') with $m'' + \chi^i \sum_{\xi\in\Xi''} p_\xi < w$ and $U^i(\Xi'',m'') \geq u$. Then, we have that $m'' < w - \chi^i \sum_{\xi\in\Xi''} p_\xi$. By Assumption 4, it follows that

$$U^i\left(\Xi'', w - \chi^i \sum_{\xi\in\Xi''} p_\xi\right) > u$$

—contradicting the definition of u. Hence, we can conclude that (C.11) must hold. Since $U^i(\Xi',m') = u$ and $m' + \chi^i \sum_{\xi\in\Xi'} p_\xi = w$, it follows that $\Xi' \in D^i_{\mathrm{H}}(\mathbf{p};u)$. Since

$\Xi' \in D^i_{\text{M}}(\mathbf{p}, w)$ was arbitrary, we can also conclude that $D^i_{\text{M}}(\mathbf{p}, w) \subseteq D^i_{\text{H}}(\mathbf{p}; u)$.

To complete the argument, we show that $D^i_{\text{H}}(\mathbf{p}; u) \supseteq D^i_{\text{M}}(\mathbf{p}, w)$. Let $\Xi'' \in D^i_{\text{H}}(\mathbf{p}; u)$ be arbitrary, and let m'' be such that

$$(\Xi'', m'') \in \underset{(\Xi,m)|U^i(\Xi,m)\geq u}{\arg\min} \left\{ m + \chi^i \sum_{\xi\in\Xi} p_\xi \right\}.$$

We have that $U^i(\Xi'', m'') \geq u$. And by (C.11), we have that $m'' = w - \chi^i \sum_{\xi\in\Xi''} p_\xi$. By the definition of u, it follows that $\Xi'' \in D^i_{\text{M}}(\mathbf{p}, w)$. Since $\Xi'' \in D^i_{\text{H}}(\mathbf{p}; w)$ was arbitrary, we can also conclude that $D^i_{\text{M}}(\mathbf{p}, w) \supseteq D^i_{\text{H}}(\mathbf{p}; u)$.

It follows that $D^i_{\text{M}}(\mathbf{p}, w) = D^i_{\text{H}}(\mathbf{p}; u)$—as claimed.

Proof of Part (b). Taking $\Xi'' \in D^i_{\text{H}}(\mathbf{p}; u)$, and letting m'' be such that

$$(\Xi'', m'') \in \underset{(\Xi,m)|U^i(\Xi,m)\geq u}{\arg\min} \left\{ m + \chi^i \sum_{\xi\in\Xi} p_\xi \right\},$$

we have that $m'' + \chi^i \sum_{\xi\in\Xi''} p_\xi = w$ and that $U^i(\Xi'', m'') \geq u$ by construction. It follows that

$$u \leq \max_{\Xi\subseteq\Omega_i} U^i \left(\Xi, m - \chi^i \sum_{\xi\in\Xi} p_\xi \right). \tag{C.12}$$

Next, let $\Xi' \in D^i_{\text{M}}(\mathbf{p}, w)$. Letting $m' = w - \chi^i \sum_{\xi\in\Xi'} p_\xi$, it follows from (C.12) that $U^i(\Xi', m') \geq u$. Since $w = m' + \chi^i \sum_{\xi\in\Xi'} p_\xi$, it thus follows from the definition of w that $\Xi' \in D^i_{\text{H}}(\mathbf{p}; u)$. Since $\Xi' \in D^i_{\text{M}}(\mathbf{p}, w)$ was arbitrary, we can conclude that $D^i_{\text{M}}(\mathbf{p}, w) \subseteq D^i_{\text{H}}(\mathbf{p}; u)$.

Now suppose that i is a seller, or that i is a buyer and w is a sufficient income for i. We claim that $\Xi'' \in D^i_{\text{M}}(\mathbf{p}, w)$. Suppose for sake of deriving a contradiction that $\Xi'' \notin D^i_{\text{M}}(\mathbf{p}, w)$. In this case, as $m'' = w - \chi^i \sum_{\xi\in\Xi''} p_\xi w$ and $U^i(\Xi'', m'') \geq u$, we must have that $U^i(\Xi', m') > u$. We show that there exists $\varepsilon > 0$ such that $U^i(\Xi', m' - \varepsilon) \geq u$ by dividing into cases based on whether i is a seller or a buyer.

Case 1: i is a seller. In this case, the existence of such a ε follows from Assumptions 1 and 3.

Case 2: i is a buyer and w is a sufficient income for i. In this case, note that $U^i(\varnothing, w) \leq U^i(\Xi', m')$ must hold by the definition of Ξ'. Hence, the sufficiency of

w as an income implies that $m' > \underline{m}^i$. Therefore, the existence of an ε satisfying the desired condition follows from Assumptions 2 and 3.

But $m' - \varepsilon + \chi^i \sum_{\xi \in \Xi} p_\xi < w$—which contradicts the definition of w as the minimum expenditure needed to obtain utility at least u since $U^i(\Xi', m' - \varepsilon) \geq u$. Hence, we can conclude that $\Xi'' \in D^i_{\mathrm{M}}(\mathbf{p}, w)$. Since $\Xi' \in D^i_{\mathrm{M}}(\mathbf{p}, w)$ was arbitrary, it follows that $D^i_{\mathrm{M}}(\mathbf{p}, w) \subseteq D^i_{\mathrm{H}}(\mathbf{p}; u)$.

D Additional details for the examples

D.1 Net substitutability in Example 4

To show that b's utility function is net substitutable, we use the approach described in Appendix B and characterize b's Hicksian valuations. The definition of Hicksian valuations entail that

$$V^b_{\mathrm{H}}(\Xi; u) = \min\{V^b(\Xi) - u, 0\}.$$

And since valuations on the right-hand side are substitutable for all u, b's utility function is net substitutable.

D.2 Net substitutability in Example 9

To show that b's utility function is net substitutable, we use the approach described in Appendix B and characterize b's Hicksian valuations. The definition of Hicksian valuations entails that

$$V^b_{\mathrm{H}}(\Xi; u) = \begin{cases} V^b(\Xi) - u & \text{if } u \leq 0 \text{ or } \zeta \notin \Xi \\ V^b(\Xi) - \frac{u}{11} & \text{if } u \geq 0 \text{ and } \zeta \in \Xi \end{cases}.$$

Since V^b is additive across trades, we see that $V^b_{\mathrm{H}}(\cdot; u)$ is additive across trades, hence in particular substitutable, for all utility levels u. It follows that U^b is net substitutable.

A similar argument shows that s's utility function is net substitutable.

D.3 Manipulability of stable mechanisms in Example 10

If s_1 reported a reservation value of $\$6 - \varepsilon$, we claim that s_1 would be matched in every stable outcome (and paid at least $\$6 - \varepsilon$). To see this, consider an outcome A in which s_1 is unmatched. Note that A cannot be individually rational if either $\hat{\omega}'$ were executed: $\hat{s}$'s reservation value for the trade exceeds b''s value for it. Since s_2's reservation value is 0, A cannot be stable if she were unmatched. If s_2 is matched to b' (i.e., ω_2' is executed) then $(\omega_1, 6 - \frac{\varepsilon}{2})$ is a block. If s_2 is matched to b (i.e., ω_2 is executed) then $(\omega_1', 6 - \frac{\varepsilon}{2})$ is a block. Hence, we can conclude that s_1 would be matched after the deviation under any stable mechanism. But then she would be paid at least $\$6 - \varepsilon$ after a deviation, and therefore has a profitable deviation in the original economy under any stable mechanism under which she is paid less than \$6.

On the other hand, if $\hat{s}$ reported a reservation value of \$5 for $\hat{\omega}$ and a reservation value of $\$\varepsilon$ for $\hat{\omega}'$ (where $\varepsilon < 0.1$), we claim that $\hat{\omega}$ would be executed (at a price of at least \$5) in every stable outcome. To complete the argument, we divide into cases to show that outcomes in which $\hat{\omega}$ is not executed must be unstable.

Case 1: Consider an outcome A in which $\hat{s}$ is unmatched. In this case, if either of s_1, s_2 remains unmatched, then there is a block involving them. If neither ω_1' nor ω_2' is executed, or one of them is executed at a price greater than $\$3 + \epsilon$, then $(\hat{\omega}', \varepsilon + \delta)$ is a block for δ sufficiently small. Hence, one of ω_1', ω_2'—say ω_1'—must be executed at a price of at most $\$3 + \varepsilon$. If ω_2 is executed at a price greater than $\$3 + \varepsilon$, then $(\omega_1, 3 + \varepsilon + \delta)$ is a block for sufficiently small δ. But if ω_2 is executed as a price of $\$3 + \varepsilon$ or less, then $(\hat{\omega}, 5 + \delta)$ is a block for a sufficiently small δ.

Case 2: Consider an outcome A in which $\hat{\omega}'$ is executed. The price of $\hat{\omega}'$ must be positive to ensure that A be individually rational for $\hat{s}$. And if A is individually rational for b, then ω_1 and ω_2 cannot both be executed at positive prices. But if ω_1 is not executed, or executed at a price of \$0, then $(\omega_1', 2)$ is a block; a symmetric argument applies if ω_2 is not executed.

Both cases are incompatible with stability, and hence we can conclude that $\hat{\omega}$ must be executed in every stable outcome after the deviation. But then $\hat{s}$ would be paid at least \$5 after a deviation, and therefore has a profitable deviation in the original economy under any stable mechanism under which she is unmatched.

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