Matching and Prices

Ravi Jagadeesan
Stanford

Alex Teytelboym
Oxford

Special Lectures at The University of Tokyo
The University of Tokyo Market Design Center (UTMD) and
Center for International Research on the Japanese Economy (CIRJE)

September 24, 2021

Motivation

- many markets involve indivisible and personalized interactions
  - auctions, labor markets, online platforms, . . .
- focus in most models: sellers' constraints on what they can sell
  - e.g., spectrum auctions: government faces interference constraints on sales
  - e.g., residency and day-care allocation (Kamada and Kojima, 2015 . . .)

Motivation

- many markets involve indivisible and personalized interactions
  - auctions, labor markets, online platforms, ...
- focus in most models: sellers' constraints on what they can sell
  - e.g., spectrum auctions: government faces interference constraints on sales
  - e.g., residency and day-care allocation (Kamada and Kojima, 2015 ...)

- buyers are often constrained in how much they can pay
  - spectrum auctions: telecom companies have limited budgets
  - labor markets: firms have hiring budgets

Motivation

- many markets involve indivisible and personalized interactions
  - auctions, labor markets, online platforms, ...
- focus in most models: sellers' constraints on what they can sell
  - e.g., spectrum auctions: government faces interference constraints on sales
  - e.g., residency and day-care allocation (Kamada and Kojima, 2015 ...)

- buyers are often constrained in how much they can pay
  - spectrum auctions: telecom companies have limited budgets
  - labor markets: firms have hiring budgets
- difference from Lecture 3: agents can exhaust their budgets in the market
  - called "(hard) budget constraint" in auction and matching literatures

Motivation

- many markets involve indivisible and personalized interactions
  - auctions, labor markets, online platforms, ...
- focus in most models: sellers' constraints on what they can sell
  - e.g., spectrum auctions: government faces interference constraints on sales
  - e.g., residency and day-care allocation (Kamada and Kojima, 2015 ...)

- buyers are often constrained in how much they can pay
  - spectrum auctions: telecom companies have limited budgets
  - labor markets: firms have hiring budgets
- difference from Lecture 3: agents can exhaust their budgets in the market
  - called "(hard) budget constraint" in auction and matching literatures
- but competitive equilibria may not exist w/indivisibilities+budget constraints

Motivation

- many markets involve indivisible and personalized interactions
  - auctions, labor markets, online platforms, . . .
- focus in most models: sellers' constraints on what they can sell
  - e.g., spectrum auctions: government faces interference constraints on sales
  - e.g., residency and day-care allocation (Kamada and Kojima, 2015 . . .)

- buyers are often constrained in how much they can pay
  - spectrum auctions: telecom companies have limited budgets
  - labor markets: firms have hiring budgets
- difference from Lecture 3: agents can exhaust their budgets in the market
  - called "(hard) budget constraint" in auction and matching literatures
- but competitive equilibria may not exist w/indivisibilities+budget constraints

- this paper exploits insights from matching theory to analyze markets for indivisible goods in which buyers can face budget constraints

Main contributions

- develop a model of two-sided, many-to-many matching with continuous transfers that allows for budget constraints as well as other income effects
- show *stable outcomes* exist if agents see interactions as *net substitutes*
  - applies even though competitive equilibria may fail to exist
- illustrate key role of flexible prices in matching markets w/budget constraints

Restrictiveness of gross substitutability with budget constraints

- key condition in most matching analyses: *gross substitutes*
  - requires, e.g., that an increase in the salary of one worker weakly raise demand for all other workers (Kelso and Crawford, 1982)
  - entails that the deferred acceptance algorithm yields a stable outcome
    - and set of stable outcomes forms a lattice, "Rural Hospitals Theorem", ...

Restrictiveness of gross substitutability with budget constraints

- key condition in most matching analyses: *gross substitutes*
  - requires, e.g., that an increase in the salary of one worker weakly raise demand for all other workers (Kelso and Crawford, 1982)
  - entails that the deferred acceptance algorithm yields a stable outcome
    - and set of stable outcomes forms a lattice, "Rural Hospitals Theorem", ...

- but gross substitutes is restrictive w/budget constraints or income effects
  - e.g., if a firm f values two workers w_1, w_2 at $\$5$ each and has $\$3$ to spend, if salaries change from $(\$1, \$2)$ to $(\$1 + \epsilon, \$2)$, then f stops demanding w_2
  - similar conclusion with income effects (when workers are not inferior)

Restrictiveness of gross substitutability with budget constraints

- key condition in most matching analyses: *gross substitutes*
  - requires, e.g., that an increase in the salary of one worker weakly raise demand for all other workers (Kelso and Crawford, 1982)
  - entails that the deferred acceptance algorithm yields a stable outcome
    - and set of stable outcomes forms a lattice, "Rural Hospitals Theorem", ...

- but gross substitutes is restrictive w/budget constraints or income effects
  - e.g., if a firm f values two workers w_1, w_2 at $\$5$ each and has $\$3$ to spend, if salaries change from $(\$1, \$2)$ to $(\$1 + \epsilon, \$2)$, then f stops demanding w_2
  - similar conclusion with income effects (when workers are not inferior)

- our approach: assume *net* substitutes instead of gross

Restrictiveness of gross substitutability with budget constraints

- key condition in most matching analyses: *gross substitutes*
  - requires, e.g., that an increase in the salary of one worker weakly raise demand for all other workers (Kelso and Crawford, 1982)
  - entails that the deferred acceptance algorithm yields a stable outcome
    - and set of stable outcomes forms a lattice, "Rural Hospitals Theorem", ...

- but gross substitutes is restrictive w/budget constraints or income effects
  - e.g., if a firm f values two workers w_1, w_2 at $\$5$ each and has $\$3$ to spend, if salaries change from $(\$1, \$2)$ to $(\$1+\epsilon, \$2)$, then f stops demanding w_2
  - similar conclusion with income effects (when workers are not inferior)

- our approach: assume *net* substitutes instead of gross
- necessitates use of topological methods instead of order-theoretic methods

Impact of price flexibility

1. price flexibility plays a key role in stability when there are budget constraints
   - w/rigid prices, stable outcomes may not exist even under net substitutes
   - unlike case of gross substitutes, where price rigidity doesn't affect stability

Impact of price flexibility

1. price flexibility plays a key role in stability when there are budget constraints
   - w/rigid prices, stable outcomes may not exist even under net substitutes
   - unlike case of gross substitutes, where price rigidity doesn't affect stability

2. price flexibility allows agents to focus on "simple" blocks
   - makes it suffice for agents to focus on "pairwise blocks" consisting of deviations between a single pair of agents (under net substitutes)
   - w/rigid prices, pairwise blocks are insufficient (even under net substitutes)
   - unlike case of gross substitutes, where pairwise blocks are always sufficient

Impact of price flexibility

1. price flexibility plays a key role in stability when there are budget constraints
   - w/rigid prices, stable outcomes may not exist even under net substitutes
   - unlike case of gross substitutes, where price rigidity doesn't affect stability

2. price flexibility allows agents to focus on "simple" blocks
   - makes it suffice for agents to focus on "pairwise blocks" consisting of deviations between a single pair of agents (under net substitutes)
   - w/rigid prices, pairwise blocks are insufficient (even under net substitutes)
   - unlike case of gross substitutes, where pairwise blocks are always sufficient

3. price flexibility is (unsurprisingly) also important to efficiency
   - ensures stable outcomes are weakly Pareto-efficient (under net substitutes)

Related literature

- matching with (continuous) transfers and income effects
  - unit-demand: Demange and Gale (1985), Kelso and Crawford (1982), ...
  - gross substitutability: Fleiner, J., Jankó, and T. (2019)
  - housing market: Quinzii (1984), Gale (1984), Svensson (1984), ...
- counterexamples with budget constraints: Mongell and Roth (1986)
- existence of equilibrium w/income effects and indivisibilities
  - Danilov, Koshevoy, and Murota (2001), and Lecture 3
- topological fixed-point methods in matching (large markets)
  - Azevedo and Hatfield (2018), Che, Kim, and Kojima (2019), Greinecker and Kah (2021), J. and Vocke (2021)

Outline

1. model
2. demand and substitutability
3. nonexistence of competitive equilibrium
4. existence of stable outcomes
5. proof of existence
6. properties of stable outcomes

Outline

1. model
2. demand and substitutability
3. nonexistence of competitive equilibrium
4. existence of stable outcomes
5. proof of existence
6. properties of stable outcomes

Model setup

- finite sets B of buyers and S of sellers

- for $s \in S$ and $b \in B$, finite set $\Omega_{s,b}$ of **trades** between s and b

$$\omega \in \Omega_{s,b}: \qquad s \xrightarrow{\omega} b$$

  - e.g., job contract without salary (Kelso and Crawford, 1982)
  - e.g., good and counterparties of exchange (Gul and Stacchetti, 1999)

Model setup

- finite sets B of buyers and S of sellers
- for $s \in S$ and $b \in B$, finite set $\Omega_{s,b}$ of **trades** between s and b

$$\omega \in \Omega_{s,b}: \qquad s \xrightarrow{\omega} b$$

    - e.g., job contract without salary (Kelso and Crawford, 1982)
    - e.g., good and counterparties of exchange (Gul and Stacchetti, 1999)
- Ω = set of all trades; Ω_j = set of trades that involve agent j

Model setup

- finite sets B of buyers and S of sellers
- for $s \in S$ and $b \in B$, finite set $\Omega_{s,b}$ of **trades** between s and b

$$\omega \in \Omega_{s,b}: \qquad s \xrightarrow{\omega} b$$

  - e.g., job contract without salary (Kelso and Crawford, 1982)
  - e.g., good and counterparties of exchange (Gul and Stacchetti, 1999)
- $\Omega =$ set of all trades; $\Omega_j =$ set of trades that involve agent j
- allow constraints on what sellers can sell, how much buyers can pay ...

Preferences

for each agent $j \in B \cup S$, there is a utility function $U^j : \mathcal{P}(\Omega_j) \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

- interpretation of level $-\infty$: violation of feasibility constraints

Preferences

for each agent $j \in B \cup S$, there is a utility function $U^j : \mathcal{P}(\Omega_j) \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

- interpretation of level $-\infty$: violation of feasibility constraints

need five technical assumptions on utility functions

Preferences

for each agent $j \in B \cup S$, there is a utility function $U^j : \mathcal{P}(\Omega_j) \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

- interpretation of level $-\infty$: violation of feasibility constraints

need five technical assumptions on utility functions

to ensure that sellers have only constraints on what they can sell, assume:

1. for each seller s, there is a family $\mathcal{F}^s \ni \varnothing$ of sets of trades that are *feasible* for s such that $U^s(\Xi, m) \in \mathbb{R}$ for $\Xi \in \mathcal{F}^s$ and $U^s(\Xi, m) = -\infty$ for $\Xi \notin \mathcal{F}^s$

Preferences

for each agent $j \in B \cup S$, there is a utility function $U^j : \mathcal{P}(\Omega_j) \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

- interpretation of level $-\infty$: violation of feasibility constraints

need five technical assumptions on utility functions

to ensure that sellers have only constraints on what they can sell, assume:

1. for each seller s, there is a family $\mathcal{F}^s \ni \varnothing$ of sets of trades that are *feasible* for s such that $U^s(\Xi, m) \in \mathbb{R}$ for $\Xi \in \mathcal{F}^s$ and $U^s(\Xi, m) = -\infty$ for $\Xi \notin \mathcal{F}^s$

to ensure that buyers only have constraints on how much they can pay, assume:

2. for each buyer b, there is a lower bound $\underline{m}^b \in \mathbb{R} \cup \{-\infty\}$ on consumption of money such that $U^b(\Xi, m) \in \mathbb{R}$ for $m > \underline{m}^b$, and $U^b(\Xi, m) = -\infty$ for $m < \underline{m}^b$

Preferences (II)

two standard conditions: continuity and monotonicity

3. [continuity] all agents' utility functions are continuous in money away from level $-\infty$. and for all buyers b, and sets $\Xi \subseteq \Omega_b$ of trades

$$\lim_{m \to (\underline{m}^b)^+} U^b(\Xi, m) = U^b(\Xi, \underline{m}^b),$$

   where we write $U^b(\Xi, -\infty) = -\infty$

4. [monotonicity] away from utility level $-\infty$, all agents' utility functions are strictly increasing in money, and buyers' (resp. sellers') utility functions are weakly increasing (resp. weakly decreasing) in trades

Preferences (II)

two standard conditions: continuity and monotonicity

3. [continuity] all agents' utility functions are continuous in money away from level $-\infty$. and for all buyers b, and sets $\Xi \subseteq \Omega_b$ of trades

$$\lim_{m \to (\underline{m}^b)^+} U^b(\Xi, m) = U^b(\Xi, \underline{m}^b),$$

   where we write $U^b(\Xi, -\infty) = -\infty$

4. [monotonicity] away from utility level $-\infty$, all agents' utility functions are strictly increasing in money, and buyers' (resp. sellers') utility functions are weakly increasing (resp. weakly decreasing) in trades

one innocuous assumption to ensure that "Hicksian valuations" are well-behaved:

5. for all sellers s, we have $\lim_{m \to -\infty} U^s(\varnothing, m) = -\infty$, and for all buyers b, we have $\lim_{m \to \infty} U^b(\varnothing, m) = \infty$

Examples of preferences

example (quasilinear utility "without a budget constraint")

$U^j(\Xi, m) = V^j(\Xi) + m$. here, $\underline{m}^j = -\infty$

Examples of preferences

example (quasilinear utility "without a budget constraint")

$U^j(\Xi, m) = V^j(\Xi) + m$. here, $\underline{m}^j = -\infty$

example (quasilinear utility "with a hard budget constraint")

$U^b(\Xi, m) = \begin{cases} V^b(\Xi) + m & \text{if } m \geq 0 \\ -\infty & \text{if } m < 0 \end{cases}$. here, $\underline{m}^b = 0$, and can run out of money

Examples of preferences

example (quasilinear utility "without a budget constraint")

$U^j(\Xi, m) = V^j(\Xi) + m$. here, $\underline{m}^j = -\infty$

example (quasilinear utility "with a hard budget constraint")

$$U^b(\Xi, m) = \begin{cases} V^b(\Xi) + m & \text{if } m \geq 0 \\ -\infty & \text{if } m < 0 \end{cases}.$$ here, $\underline{m}^b = 0$, and can run out of money

example (quasilogarithmic utility—Lecture 3)

$$U^b(\Xi, m) = \begin{cases} \log(m) - \log(-V^b_{\mathrm{Q}}(\Xi)) & \text{if } m > 0 \\ -\infty & \text{if } m \leq 0 \end{cases}.$$ here, $\underline{m}^b = 0$, but can't run out

Outline

1. model
2. demand and substitutability
3. nonexistence of competitive equilibrium
4. existence of stable outcomes
5. proof of existence
6. properties of stable outcomes

Marshallian and Hicksian demand for trades

- Marshallian: for a buyer b, an *income* $w > \underline{m}^b$, and a price vector $\mathbf{p} \in \mathbb{R}^\Omega$, let

$$D^b_{\mathrm{M}}(\mathbf{p}, w) = \left\{ \Xi^* \middle| (\Xi^*, m^*) \text{ maximizes } U^b(\Xi, m) \text{ subject to } m + \sum_{\xi \in \Xi} p_\xi \leq w \right\}$$

Marshallian and Hicksian demand for trades

- Marshallian: for a buyer b, an *income* $w > \underline{m}^b$, and a price vector $\mathbf{p} \in \mathbb{R}^\Omega$, let

$$D^b_{\mathrm{M}}(\mathbf{p}, w) = \left\{ \Xi^* \,\middle|\, (\Xi^*, m^*) \text{ maximizes } U^b(\Xi, m) \text{ subject to } m + \sum_{\xi \in \Xi} p_\xi \le w \right\}$$

- Hicksian: for a buyer b, a utility level u, and a price vector $\mathbf{p} \in \mathbb{R}^\Omega$, let

$$D^b_{\mathrm{H}}(\mathbf{p}; u) = \left\{ \Xi^* \,\middle|\, (\Xi^*, m^*) \text{ minimizes } m + \sum_{\xi \in \Xi} p_\xi \text{ subject to } U^b(\Xi, m) \ge u \right\}$$

Marshallian and Hicksian demand for trades

- Marshallian: for a buyer b, an *income* $w > \underline{m}^b$, and a price vector $\mathbf{p} \in \mathbb{R}^\Omega$, let

$$D^b_{\mathrm{M}}(\mathbf{p}, w) = \left\{ \Xi^* \middle| (\Xi^*, m^*) \text{ maximizes } U^b(\Xi, m) \text{ subject to } m + \sum_{\xi \in \Xi} p_\xi \leq w \right\}$$

- Hicksian: for a buyer b, a utility level u, and a price vector $\mathbf{p} \in \mathbb{R}^\Omega$, let

$$D^b_{\mathrm{H}}(\mathbf{p}; u) = \left\{ \Xi^* \middle| (\Xi^*, m^*) \text{ minimizes } m + \sum_{\xi \in \Xi} p_\xi \text{ subject to } U^b(\Xi, m) \geq u \right\}$$

- similar definitions with opposite signs on transfers for sellers
- as in Lecture 3, Hicksian demand at a utility level has a quasilinear representation as demand for a "Hicksian valuation"

Marshallian vs. Hicksian demand with budget constraints

- w/o hard budget constraints: Marshallian and Hicksian demand are upper hemicontinuous, and give the same demand sets “in optimum”
- w/hard budget constraints: Hicksian still well-behaved, but Marshallian isn’t

Marshallian vs. Hicksian demand with budget constraints

- w/o hard budget constraints: Marshallian and Hicksian demand are upper hemicontinuous, and give the same demand sets "in optimum"
- w/hard budget constraints: Hicksian still well-behaved, but Marshallian isn't

example (difference between Marshallian and Hicksian demands)

- consider a buyer b who values trade ω at $\$2$ but is budget-constrained
- if $p_\omega = 1$, two bundles cost $\$1$ and deliver utility ≥ 1: namely $(\varnothing, 1)$, $(\{\omega\}, 0)$

Marshallian vs. Hicksian demand with budget constraints

- w/o hard budget constraints: Marshallian and Hicksian demand are upper hemicontinuous, and give the same demand sets "in optimum"
- w/hard budget constraints: Hicksian still well-behaved, but Marshallian isn't

example (difference between Marshallian and Hicksian demands)

- consider a buyer b who values trade ω at $\$2$ but is budget-constrained
- if $p_\omega = 1$, two bundles cost $\$1$ and deliver utility ≥ 1: namely $(\varnothing, 1)$, $(\{\omega\}, 0)$
- the bundles deliver different utilities: $U^b(\varnothing, 1) = 1$ but $U^b(\{\omega\}, 0) = 2 \implies$

$$D^b_{\mathrm{M}}(\mathbf{p}, 1) = \{\{\omega\}\} \text{ but } D^b_{\mathrm{H}}(\mathbf{p}; 1) = \{\varnothing, \{\omega\}\}$$

Marshallian vs. Hicksian demand with budget constraints

- w/o hard budget constraints: Marshallian and Hicksian demand are upper hemicontinuous, and give the same demand sets "in optimum"
- w/hard budget constraints: Hicksian still well-behaved, but Marshallian isn't

example (difference between Marshallian and Hicksian demands)

- consider a buyer b who values trade ω at \$2 but is budget-constrained
- if $p_\omega = 1$, two bundles cost \$1 and deliver utility ≥ 1: namely $(\varnothing, 1)$, $(\{\omega\}, 0)$
- the bundles deliver different utilities: $U^b(\varnothing, 1) = 1$ but $U^b(\{\omega\}, 0) = 2 \implies$

$$D^b_{\mathrm{M}}(\mathbf{p}, 1) = \{\{\omega\}\} \text{ but } D^b_{\mathrm{H}}(\mathbf{p}; 1) = \{\varnothing, \{\omega\}\}$$

- Marshallian demand is also not upper hemicontinuous: $D^b_{\mathrm{M}}(\mathbf{p} + \epsilon, 1) = \{\varnothing\}$

Marshallian vs. Hicksian demand with budget constraints

- w/o hard budget constraints: Marshallian and Hicksian demand are upper hemicontinuous, and give the same demand sets "in optimum"
- w/hard budget constraints: Hicksian still well-behaved, but Marshallian isn't

example (difference between Marshallian and Hicksian demands)

- consider a buyer b who values trade ω at \$2 but is budget-constrained
- if $p_\omega = 1$, two bundles cost \$1 and deliver utility ≥ 1: namely $(\varnothing, 1)$, $(\{\omega\}, 0)$
- the bundles deliver different utilities: $U^b(\varnothing, 1) = 1$ but $U^b(\{\omega\}, 0) = 2 \implies$

$$D^b_{\mathrm{M}}(\mathbf{p}, 1) = \{\{\omega\}\} \text{ but } D^b_{\mathrm{H}}(\mathbf{p}; 1) = \{\varnothing, \{\omega\}\}$$

- Marshallian demand is also not upper hemicontinuous: $D^b_{\mathrm{M}}(\mathbf{p} + \epsilon, 1) = \{\varnothing\}$

- causes technical challenges when looking from the Marshallian side

Substitutes conditions

- gross substitutes is a condition on both substitution and income effects

definition (~Kelso and Crawford, 1982)

U^b is **gross substitutable at income** w if for all trades ω, price vectors $\mathbf{p}$, and prices $p'_\omega > p_\omega$ such that $D^b_{\mathrm{M}}(\mathbf{p}, w) = \{\Xi\}$ and $D^b_{\mathrm{M}}((p'_\omega, \mathbf{p}_{-\omega}), w) = \{\Xi'\}$, if $\psi \in \Xi$ and $\psi \neq \omega$, then $\psi \in \Xi'$

Substitutes conditions

- gross substitutes is a condition on both substitution and income effects

definition (~Kelso and Crawford, 1982)

U^b is **gross substitutable at income** w if for all trades ω, price vectors $\mathbf{p}$, and prices $p'_\omega > p_\omega$ such that $D^b_{\mathrm{M}}(\mathbf{p}, w) = \{\Xi\}$ and $D^b_{\mathrm{M}}((p'_\omega, \mathbf{p}_{-\omega}), w) = \{\Xi'\}$, if $\psi \in \Xi$ and $\psi \neq \omega$, then $\psi \in \Xi'$

- net substitutes is an analogous condition on substitution effects alone

definition (Lecture 3)

U^b is **net substitutable** if for all utility levels u, trades ω, price vectors $\mathbf{p}$, and prices $p'_\omega > p_\omega$ such that $D^b_{\mathrm{H}}(\mathbf{p}; u) = \{\Xi\}$ and $D^b_{\mathrm{H}}((p'_\omega, \mathbf{p}_{-\omega}); u) = \{\Xi'\}$, if $\psi \in \Xi$ and $\psi \neq \omega$, then $\psi \in \Xi'$

Substitutes conditions

- gross substitutes is a condition on both substitution and income effects

definition (~Kelso and Crawford, 1982)

U^b is **gross substitutable at income** w if for all trades ω, price vectors $\mathbf{p}$, and prices $p'_\omega > p_\omega$ such that $D^b_{\mathrm{M}}(\mathbf{p}, w) = \{\Xi\}$ and $D^b_{\mathrm{M}}((p'_\omega, \mathbf{p}_{-\omega}), w) = \{\Xi'\}$, if $\psi \in \Xi$ and $\psi \neq \omega$, then $\psi \in \Xi'$

- net substitutes is an analogous condition on substitution effects alone

definition (Lecture 3)

U^b is **net substitutable** if for all utility levels u, trades ω, price vectors $\mathbf{p}$, and prices $p'_\omega > p_\omega$ such that $D^b_{\mathrm{H}}(\mathbf{p}; u) = \{\Xi\}$ and $D^b_{\mathrm{H}}((p'_\omega, \mathbf{p}_{-\omega}); u) = \{\Xi'\}$, if $\psi \in \Xi$ and $\psi \neq \omega$, then $\psi \in \Xi'$

- equivalent conditions under quasilinearity (without budget constraints)

Budget constraints and gross vs. net substitutes

example (budget constraints and failure of gross substitutes)

- consider a buyer b who has a quasilinear utility function with a hard budget constraint and values trades ω_1, ω_2 at $\$5$ each
- with an income of $w = \$3$, utility function U^b is not gross substitutable

Budget constraints and gross vs. net substitutes

example (budget constraints and failure of gross substitutes)

- consider a buyer b who has a quasilinear utility function with a hard budget constraint and values trades ω_1, ω_2 at $\$5$ each
- with an income of $w = \$3$, utility function U^b is not gross substitutable
- but it is net substitutable (proof using "Hicksian valuations" in the paper)

Budget constraints and gross vs. net substitutes

example (budget constraints and failure of gross substitutes)

- consider a buyer b who has a quasilinear utility function with a hard budget constraint and values trades ω_1, ω_2 at $\$5$ each
- with an income of $w = \$3$, utility function U^b is not gross substitutable
- but it is net substitutable (proof using "Hicksian valuations" in the paper)

proposition

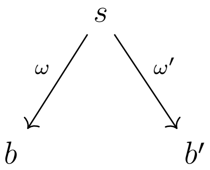
if U^j is gross substitutable at all incomes, and strictly increasing in trades away from utility level $-\infty$ if j is a buyer, then U^j is net substitutable

- shown without budget constraints in Lecture 3

Outline

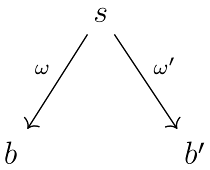
1. model
2. demand and substitutability
3. nonexistence of competitive equilibrium
4. existence of stable outcomes
5. proof of existence
6. properties of stable outcomes

Nonexistence of competitive equilibria w/budget constraints (I)



- if s is only willing to engage in one trade, and has reservation value 0, and each buyer values trade at $\$2$ but has an income of only $\$1$, then there are no competitive equilibria
  - both buyers demand trade if price $\leq \$1$; neither if price $> \$1$

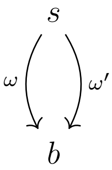
Nonexistence of competitive equilibria w/budget constraints (I)



- if s is only willing to engage in one trade, and has reservation value 0, and each buyer values trade at $\$2$ but has an income of only $\$1$, then there are no competitive equilibria
  - both buyers demand trade if price $\leq \$1$; neither if price $> \$1$

- example may seem knife-edge, but phenomenon is more general . . .

Nonexistence of competitive equilibria w/budget constraints (II)



- if s is only willing to engage in one trade, and has reservation value 0, and

$$U^b(\Xi, m) = \begin{cases} m & \text{if } m \geq 0 \text{ and } \Xi = \varnothing \\ m + \min\{m, 1\} & \text{if } m \geq 0 \text{ and } |\Xi| = 1 \\ m + 1 + \min\{m, 1\} & \text{if } m \geq 0 \text{ and } |\Xi| = 2 \\ -\infty & \text{if } m < 0 \end{cases},$$

  then there are no competitive equilibria if $w^b < 1$
  - b demands both trades if price $\leq \frac{w^b}{2}$; neither if price $> \frac{w^b}{2}$

Outline

1. model
2. demand and substitutability
3. nonexistence of competitive equilibrium
4. existence of stable outcomes
5. proof of existence
6. properties of stable outcomes

Stable outcomes (I)

- since competitive equilibria may not exist, consider instead *stable outcomes*

Stable outcomes (I)

- since competitive equilibria may not exist, consider instead *stable outcomes*
- a **contract** is a pair (ω, p_ω), where $p_\omega \in \mathbb{R}$
  - e.g., job contract with salary (Kelso and Crawford, 1982; Roth, 1984)

Stable outcomes (I)

- since competitive equilibria may not exist, consider instead *stable outcomes*

- a **contract** is a pair (ω, p_ω), where $p_\omega \in \mathbb{R}$
  - e.g., job contract with salary (Kelso and Crawford, 1982; Roth, 1984)
- an **outcome** is a set of contracts that contains at most price for each trade
  - e.g., in a labor market, a matching of workers to firms, and salaries

Stable outcomes (I)

- since competitive equilibria may not exist, consider instead *stable outcomes*
- a **contract** is a pair (ω, p_ω), where $p_\omega \in \mathbb{R}$
  - e.g., job contract with salary (Kelso and Crawford, 1982; Roth, 1984)
- an **outcome** is a set of contracts that contains at most price for each trade
  - e.g., in a labor market, a matching of workers to firms, and salaries
- given a set Y of contracts involving an agent j and an income w, define j's **choice correspondence** by

$$C^j(Y, w) = \underset{\text{outcomes } Z \subseteq Y}{\arg\max} \; U^j(\text{trades in } Z, \text{ending money balance if } Z \text{ executed})$$

Stable outcomes (II)

(Roth, 1984; Hatfield and Milgrom 2005; Hatfield et al., 2013)

definition

given an income profile $(w^j)_{j \in B \cup S}$, an outcome A is:

- **individually rational** if $A_j \in C^j(A_j, w^j)$ for all j

Stable outcomes (II)

(Roth, 1984; Hatfield and Milgrom 2005; Hatfield et al., 2013)

definition

given an income profile $(w^j)_{j \in B \cup S}$, an outcome A is:

- **individually rational** if $A_j \in C^j(A_j, w^j)$ for all j
- **blocked** by a nonempty set Z of contracts if for all $W \in C^j(A_j \cup Z_j, w^j)$, we have that $W \supseteq Z_j$

Stable outcomes (II)

(Roth, 1984; Hatfield and Milgrom 2005; Hatfield et al., 2013)

definition

given an income profile $(w^j)_{j \in B \cup S}$, an outcome A is:

- **individually rational** if $A_j \in C^j(A_j, w^j)$ for all j
- **blocked** by a nonempty set Z of contracts if for all $W \in C^j(A_j \cup Z_j, w^j)$, we have that $W \supseteq Z_j$
- **stable** if individually rational and unblocked

Stable outcomes (II)

(Roth, 1984; Hatfield and Milgrom 2005; Hatfield et al., 2013)

definition

given an income profile $(w^j)_{j \in B \cup S}$, an outcome A is:

- **individually rational** if $A_j \in C^j(A_j, w^j)$ for all j
- **blocked** by a nonempty set Z of contracts if for all $W \in C^j(A_j \cup Z_j, w^j)$, we have that $W \supseteq Z_j$
- **stable** if individually rational and unblocked

differs from core in three ways

Stable outcomes (II)

(Roth, 1984; Hatfield and Milgrom 2005; Hatfield et al., 2013)

definition

given an income profile $(w^j)_{j \in B \cup S}$, an outcome A is:

- **individually rational** if $A_j \in C^j(A_j, w^j)$ for all j
- **blocked** by a nonempty set Z of contracts if for all $W \in C^j(A_j \cup Z_j, w^j)$, we have that $W \supseteq Z_j$
- **stable** if individually rational and unblocked

differs from core in three ways

1. imposition of individual rationality

Stable outcomes (II)

(Roth, 1984; Hatfield and Milgrom 2005; Hatfield et al., 2013)

definition

given an income profile $(w^j)_{j \in B \cup S}$, an outcome A is:

- **individually rational** if $A_j \in C^j(A_j, w^j)$ for all j
- **blocked** by a nonempty set Z of contracts if for all $W \in C^j(A_j \cup Z_j, w^j)$, we have that $W \supseteq Z_j$
- **stable** if individually rational and unblocked

differs from core in three ways

1. imposition of individual rationality
2. agents in a blocking coalition can retain existing contracts with outsiders

Stable outcomes (II)

(Roth, 1984; Hatfield and Milgrom 2005; Hatfield et al., 2013)

definition

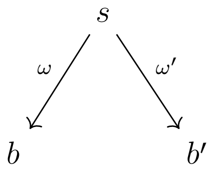
given an income profile $(w^j)_{j \in B \cup S}$, an outcome A is:

- **individually rational** if $A_j \in C^j(A_j, w^j)$ for all j
- **blocked** by a nonempty set Z of contracts if for all $W \in C^j(A_j \cup Z_j, w^j)$, we have that $W \supseteq Z_j$
- **stable** if individually rational and unblocked

differs from core in three ways

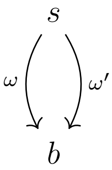
1. imposition of individual rationality
2. agents in a blocking coalition can retain existing contracts with outsiders
3. agents in a blocking coalition must want to choose all blocking contracts (rather than merely get a utility improvement)

Competitive equilibrium vs. stable outcomes



- if s is only willing to engage in one trade, and has reservation value 0, and each buyer values trade at $\$2$ but has an income of only $\$1$, then there are no competitive equilibria
  - both buyers demand trade if price $\leq \$1$; neither if price $> \$1$
- but there are stable outcomes, in which one buyer buys at a price of $\$1$
  - the other buyer is unhappy, but can't make the seller a better offer

Competitive equilibrium vs. stable outcomes (II)



- if s is only willing to engage in one trade, and has reservation value 0, and

$$U^b(\Xi, m) = \begin{cases} m & \text{if } m \geq 0 \text{ and } \Xi = \varnothing \\ m + \min\{m, 1\} & \text{if } m \geq 0 \text{ and } |\Xi| = 1 \\ m + 1 + \min\{m, 1\} & \text{if } m \geq 0 \text{ and } |\Xi| = 2 \\ -\infty & \text{if } m < 0 \end{cases}$$

then for $w^b < 1$, there are stable outcomes: one trade executed at price $\frac{w^b}{2}$

- other trade doesn't give a block since b can't offer more than $\frac{w^b}{2}$ for it

Existence of stable outcomes

theorem

under net substitutes, stable outcomes exist for all income profiles

- generalizes previous existence results for matching with transfers that assume quasilinearity or gross substitutability
  - Crawford and Knoer (1981), Kelso and Crawford (1982);
    two-sided versions of Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), Fleiner, J., Jankó, and T. (2019)

Existence of stable outcomes

theorem

under net substitutes, stable outcomes exist for all income profiles

- generalizes previous existence results for matching with transfers that assume quasilinearity or gross substitutability
  - Crawford and Knoer (1981), Kelso and Crawford (1982); two-sided versions of Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), Fleiner, J., Jankó, and T. (2019)
- but unlike those results, flexibility of prices plays a critical role
  - w/o flexible prices: trades with different counterparties "must" (in a maximal domain sense) be gross substitutes for stable outcomes to exist
    - shown by Hatfield and Kojima (2008) for many-to-one matching

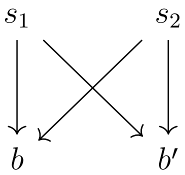
Existence of stable outcomes

theorem

under net substitutes, stable outcomes exist for all income profiles

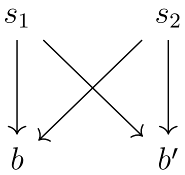
- generalizes previous existence results for matching with transfers that assume quasilinearity or gross substitutability
  - Crawford and Knoer (1981), Kelso and Crawford (1982); two-sided versions of Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), Fleiner, J., Jankó, and T. (2019)
- but unlike those results, flexibility of prices plays a critical role
  - w/o flexible prices: trades with different counterparties "must" (in a maximal domain sense) be gross substitutes for stable outcomes to exist
    - shown by Hatfield and Kojima (2008) for many-to-one matching
  - net substitutes permits gross complementarities between all trades, so is weaker than Hatfield and Kojima's (2008) maximal domain condition

Role of price flexibility



- many-to-one market w/2 sellers and 2 buyers
- b has quasilogarithmic utility w/additive quasivaluation
- b' has unit-demand and prefers s_2 by $\$1$
- s_1 prefers to work for b' by $\$1$; s_2 prefers b by $\$1$

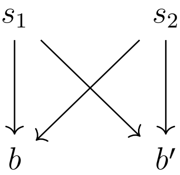
Role of price flexibility



- many-to-one market w/2 sellers and 2 buyers
- b has quasilogarithmic utility w/additive quasivaluation
- b' has unit-demand and prefers s_2 by $\$1$
- s_1 prefers to work for b' by $\$1$; s_2 prefers b by $\$1$
- say salaries were fixed at $\$4$; then preferences over sets of counterparties are

$$s_1 : \{b'\} \succ_{s_1} \{b\} \succ_{s_1} \varnothing \qquad s_2 : \{b\} \succ_{s_2} \{b'\} \succ_{s_2} \varnothing$$

Role of price flexibility

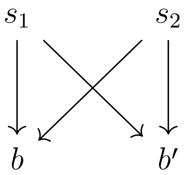


- many-to-one market w/2 sellers and 2 buyers
- b has quasilogarithmic utility w/additive quasivaluation
- b' has unit-demand and prefers s_2 by $\$1$
- s_1 prefers to work for b' by $\$1$; s_2 prefers b by $\$1$
- say salaries were fixed at $\$4$; then preferences over sets of counterparties are

$$s_1 : \{b'\} \succ_{s_1} \{b\} \succ_{s_1} \varnothing \qquad s_2 : \{b\} \succ_{s_2} \{b'\} \succ_{s_2} \varnothing$$

$$b : \{s_1, s_2\} \succ_b \{s_1\} \succ_b \varnothing \qquad b' : \{s_2\} \succ_{b'} \{s_1\} \succ_{b'} \varnothing$$

Role of price flexibility



- many-to-one market w/2 sellers and 2 buyers
- b has quasilogarithmic utility w/additive quasivaluation
- b' has unit-demand and prefers s_2 by $\$1$
- s_1 prefers to work for b' by $\$1$; s_2 prefers b by $\$1$

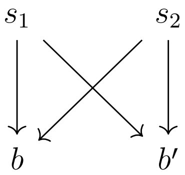
- say salaries were fixed at $\$4$; then preferences over sets of counterparties are

$$s_1 : \{b'\} \succ_{s_1} \{b\} \succ_{s_1} \varnothing \qquad s_2 : \{b\} \succ_{s_2} \{b'\} \succ_{s_2} \varnothing$$

$$b : \{s_1, s_2\} \succ_b \{s_1\} \succ_b \varnothing \qquad b' : \{s_2\} \succ_{b'} \{s_1\} \succ_{b'} \varnothing$$

$\Longrightarrow$ there would be no stable outcome (Alva, 2013)

Role of price flexibility



- many-to-one market w/2 sellers and 2 buyers
- b has quasilogarithmic utility w/additive quasivaluation
- b' has unit-demand and prefers s_2 by $\$1$
- s_1 prefers to work for b' by $\$1$; s_2 prefers b by $\$1$

- say salaries were fixed at $\$4$; then preferences over sets of counterparties are

$$s_1 : \{b'\} \succ_{s_1} \{b\} \succ_{s_1} \varnothing \qquad s_2 : \{b\} \succ_{s_2} \{b'\} \succ_{s_2} \varnothing$$

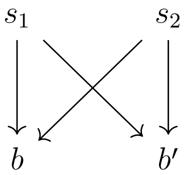
$$b : \{s_1, s_2\} \succ_b \{s_1\} \succ_b \varnothing \qquad b' : \{s_2\} \succ_{b'} \{s_1\} \succ_{b'} \varnothing$$

$\implies$ there would be no stable outcome (Alva, 2013)

- but if s_1's salary at b could rise to $\$4\frac{1}{2}$, then b's preferences would become

$$b : \{s_1\} \succ_b \{s_1, s_2\} \succ_b \varnothing$$

Role of price flexibility



- many-to-one market w/2 sellers and 2 buyers
- b has quasilogarithmic utility w/additive quasivaluation
- b' has unit-demand and prefers s_2 by \$1
- s_1 prefers to work for b' by \$1; s_2 prefers b by \$1
- say salaries were fixed at \$4; then preferences over sets of counterparties are

$$s_1 : \{b'\} \succ_{s_1} \{b\} \succ_{s_1} \varnothing \qquad s_2 : \{b\} \succ_{s_2} \{b'\} \succ_{s_2} \varnothing$$

$$b : \{s_1, s_2\} \succ_b \{s_1\} \succ_b \varnothing \qquad b' : \{s_2\} \succ_{b'} \{s_1\} \succ_{b'} \varnothing$$

$\implies$ there would be no stable outcome (Alva, 2013)

- but if s_1's salary at b could rise to $\$4\frac{1}{2}$, then b's preferences would become

$$b : \{s_1\} \succ_b \{s_1, s_2\} \succ_b \varnothing$$

$\implies$ get stable outcome in which s_1 and b matched, and s_2 and b' matched

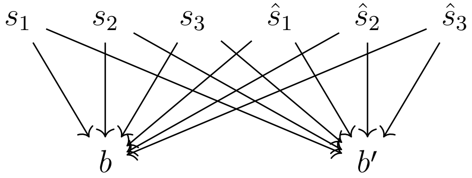
Failure of deferred acceptance

- Lecture 3: saw multi-unit ascending auctions may fail to lead to competitive equilibrium when buyers experience income effects
- here: example showing deferred acceptance may fail in a labor market when firms can experience income effects but have net substitutes preferences

Failure of deferred acceptance

- Lecture 3: saw multi-unit ascending auctions may fail to lead to competitive equilibrium when buyers experience income effects
- here: example showing deferred acceptance may fail in a labor market when firms can experience income effects but have net substitutes preferences

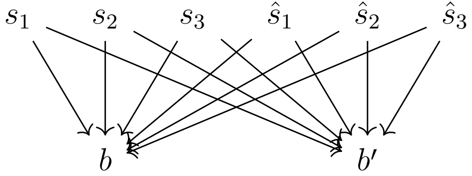
- six workers: three (s_1, s_2, s_3) of one type, three $(\hat{s}_1, \hat{s}_2, \hat{s}_3)$ of another type
- two identical firms b, b' who each want to hire up to two workers of each type



Failure of deferred acceptance

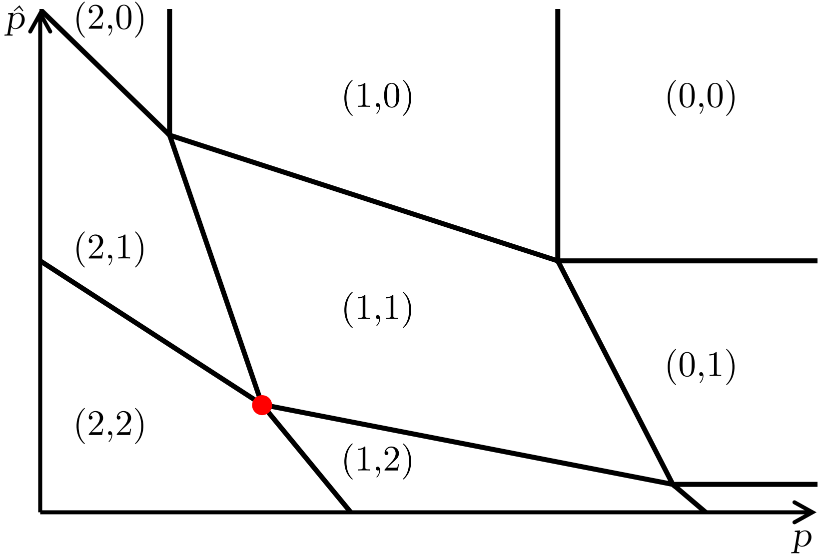
- Lecture 3: saw multi-unit ascending auctions may fail to lead to competitive equilibrium when buyers experience income effects
- here: example showing deferred acceptance may fail in a labor market when firms can experience income effects but have net substitutes preferences

- six workers: three (s_1, s_2, s_3) of one type, three $(\hat{s}_1, \hat{s}_2, \hat{s}_3)$ of another type
- two identical firms b, b' who each want to hire up to two workers of each type

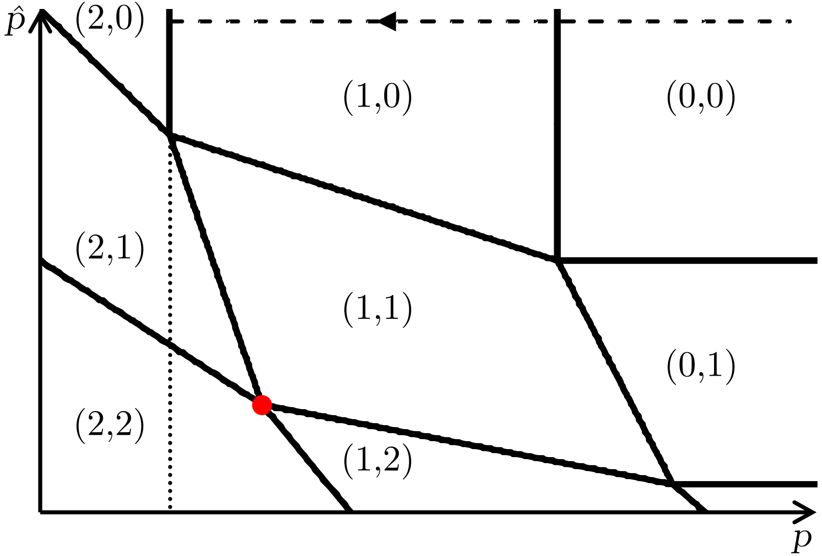


- buyers/firms have quasilogarithmic utility; sellers/workers quasilinear utility

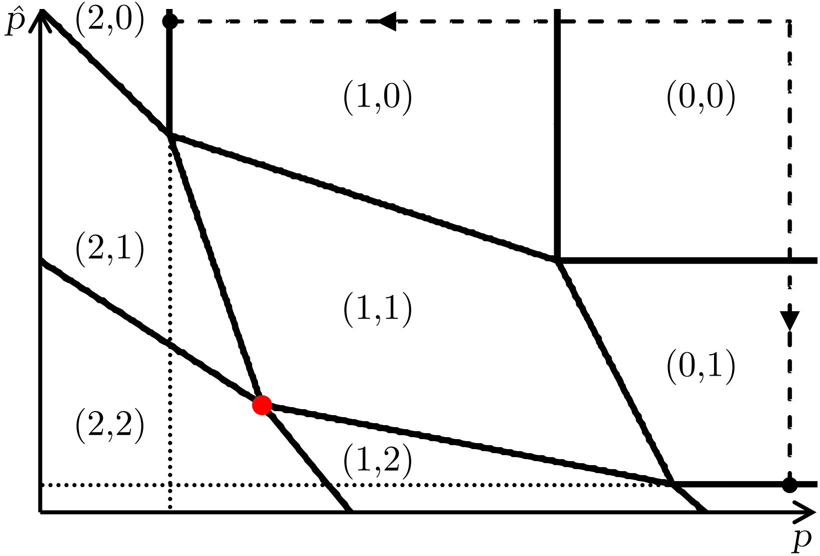
Failure of deferred acceptance



Failure of deferred acceptance



Failure of deferred acceptance



Outline

1. model
2. demand and substitutability
3. nonexistence of competitive equilibrium
4. existence of stable outcomes
5. proof of existence
6. properties of stable outcomes

Strategy of proof of existence of stable outcomes

- want to go via equilibrium existence, but can't use competitive equilibrium

Strategy of proof of existence of stable outcomes

- want to go via equilibrium existence, but can't use competitive equilibrium
- workaround (from GE): instead of requiring that agents maximize utility, require they minimize expenditure of obtaining their equilibrium utility

Strategy of proof of existence of stable outcomes

- want to go via equilibrium existence, but can't use competitive equilibrium
- workaround (from GE): instead of requiring that agents maximize utility, require they minimize expenditure of obtaining their equilibrium utility

definition

given an income profile $(w^j)_{j \in B \cup S}$, a **quasiequilibrium** consists of a set Ξ of trades and a price vector $\mathbf{p} \in \mathbb{R}^{\Omega}$

Strategy of proof of existence of stable outcomes

- want to go via equilibrium existence, but can't use competitive equilibrium
- workaround (from GE): instead of requiring that agents maximize utility, require they minimize expenditure of obtaining their equilibrium utility

definition

given an income profile $(w^j)_{j \in B \cup S}$, a **quasiequilibrium** consists of a set Ξ of trades and a price vector $\mathbf{p} \in \mathbb{R}^\Omega$ such that, writing

$$u^j = j\text{'s utility of set } \Xi_j \text{ of trades at price vector } \mathbf{p} \text{ and income } w^j,$$

we have that $u^j > -\infty$ and that $\Xi_j \in D^j_{\mathrm{H}}(\mathbf{p}; u^j)$

Strategy of proof of existence of stable outcomes

- want to go via equilibrium existence, but can't use competitive equilibrium
- workaround (from GE): instead of requiring that agents maximize utility, require they minimize expenditure of obtaining their equilibrium utility

definition

given an income profile $(w^j)_{j \in B \cup S}$, a **quasiequilibrium** consists of a set Ξ of trades and a price vector $\mathbf{p} \in \mathbb{R}^\Omega$ such that, writing

$$u^j = j\text{'s utility of set } \Xi_j \text{ of trades at price vector } \mathbf{p} \text{ and income } w^j,$$

we have that $u^j > -\infty$ and that $\Xi_j \in D^j_{\mathrm{H}}(\mathbf{p}; u^j)$

- correspond to competitive equilibria in Hicksian economies (Lecture 3)

Strategy of proof of existence of stable outcomes

- want to go via equilibrium existence, but can't use competitive equilibrium
- workaround (from GE): instead of requiring that agents maximize utility, require they minimize expenditure of obtaining their equilibrium utility

definition

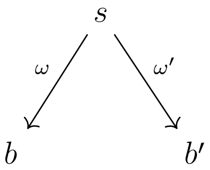
given an income profile $(w^j)_{j \in B \cup S}$, a **quasiequilibrium** consists of a set Ξ of trades and a price vector $\mathbf{p} \in \mathbb{R}^\Omega$ such that, writing

$$u^j = j\text{'s utility of set } \Xi_j \text{ of trades at price vector } \mathbf{p} \text{ and income } w^j,$$

we have that $u^j > -\infty$ and that $\Xi_j \in D^j_{\mathrm{H}}(\mathbf{p}; u^j)$

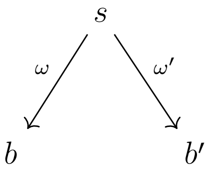
- correspond to competitive equilibria in Hicksian economies (Lecture 3)
- by "forgetting" prices of unrealized trades, obtain a **quasiequilibrium outcome** from each quasiequilibrium

Competitive equilibrium vs. quasiequilibrium



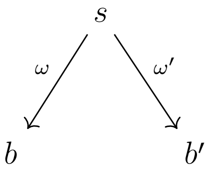
- if s is only willing to engage in one trade, and has reservation value 0, and each buyer values trade at $\$2$ but has an income of only $\$1$, then there are no competitive equilibria
  - both buyers demand trade if price $\leq \$1$; neither if price $> \$1$

Competitive equilibrium vs. quasiequilibrium



- if s is only willing to engage in one trade, and has reservation value 0, and each buyer values trade at $\$2$ but has an income of only $\$1$, then there are no competitive equilibria
  - both buyers demand trade if price $\leq \$1$; neither if price $> \$1$
- but there are quasiequilibria in which the price is $1
  - one buyer gets to purchase (gets utility 2), the other does not (gets utility 1)
  - gives a quasiequilibrium since $D^b_{\mathrm{H}}((1,1);1) = \{\varnothing, \{\omega\}\}$

Competitive equilibrium vs. quasiequilibrium



- if s is only willing to engage in one trade, and has reservation value 0, and each buyer values trade at $\$2$ but has an income of only $\$1$, then there are no competitive equilibria
  - both buyers demand trade if price $\leq \$1$; neither if price $> \$1$
- but there are quasiequilibria in which the price is $\$1$
  - one buyer gets to purchase (gets utility 2), the other does not (gets utility 1)
  - gives a quasiequilibrium since $D^b_{\mathrm{H}}((1,1);1) = \{\varnothing, \{\omega\}\}$
- and the corresponding quasiequilibrium outcomes are stable

Strategy of proof of existence of stable outcomes (II)

- using quasiequilibrium, can divide the proof into two steps

proposition

under net substitutes, quasiequilibria exist for all income profiles

proposition

every quasiequilibrium outcome is stable

Strategy of proof of existence of stable outcomes (II)

- using quasiequilibrium, can divide the proof into two steps

proposition

under net substitutes, quasiequilibria exist for all income profiles

proposition

every quasiequilibrium outcome is stable

- proof similar to the proof of the first welfare theorem
  - as adapted to matching by Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), Fleiner, J., Jankó, & T. (2019)
  - unlike most matching results, proof relies on monotonicity of utility in trades

Strategy of proof of existence of stable outcomes (II)

- using quasiequilibrium, can divide the proof into two steps

proposition

under net substitutes, quasiequilibria exist for all income profiles

- proof combines arguments from matching theory with the topological fixed point argument from proof of the Equilibrium Existence Duality (Lecture 3)

proposition

every quasiequilibrium outcome is stable

- proof similar to the proof of the first welfare theorem
  - as adapted to matching by Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), Fleiner, J., Jankó, & T. (2019)
  - unlike most matching results, proof relies on monotonicity of utility in trades

Sketch of proof of existence of quasiequilibrium

- technical difficulty relative to Lecture 3: if

$$\lim_{m\to\infty} U^s(\varnothing, m) > \lim_{m\to\infty} U^s(\Xi, m),$$

  then the Hicksian value $V^s_{\mathrm{H}}(\Xi; u)$ can be $-\infty$

Sketch of proof of existence of quasiequilibrium

- technical difficulty relative to Lecture 3: if

$$\lim_{m\to\infty} U^s(\varnothing, m) > \lim_{m\to\infty} U^s(\Xi, m),$$

  then the Hicksian value $V^s_{\mathrm{H}}(\Xi; u)$ can be $-\infty$

steps of proof:

1. apply a transformation to the economy to make Hicksian values lie in $\mathbb{R}$
   - by endowing sellers with the option to unilaterally drop a trade for a high cost (Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp, 2013)

Sketch of proof of existence of quasiequilibrium

- technical difficulty relative to Lecture 3: if

$$\lim_{m\to\infty} U^s(\varnothing, m) > \lim_{m\to\infty} U^s(\Xi, m),$$

  then the Hicksian value $V_{\mathrm{H}}^s(\Xi; u)$ can be $-\infty$

steps of proof:

1. apply a transformation to the economy to make Hicksian values lie in $\mathbb{R}$
   - by endowing sellers with the option to unilaterally drop a trade for a high cost (Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp, 2013)
2. apply topological fixed-point argument from Lecture 3 to show there is a profile $(\tilde{u}^j)_{j\in J}$ of utility levels and a competitive equilibrium in the corresponding Hicksian economy for which expenditure $=$ income

Sketch of proof of existence of quasiequilibrium

- technical difficulty relative to Lecture 3: if

$$\lim_{m\to\infty} U^s(\varnothing, m) > \lim_{m\to\infty} U^s(\Xi, m),$$

  then the Hicksian value $V_{\mathrm{H}}^s(\Xi; u)$ can be $-\infty$

steps of proof:

1. apply a transformation to the economy to make Hicksian values lie in $\mathbb{R}$
   - by endowing sellers with the option to unilaterally drop a trade for a high cost (Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp, 2013)
2. apply topological fixed-point argument from Lecture 3 to show there is a profile $(\tilde{u}^j)_{j\in J}$ of utility levels and a competitive equilibrium in the corresponding Hicksian economy for which expenditure $=$ income
3. show that the equilibrium gives a quasiequilibrium in the original economy

Outline

1. model
2. demand and substitutability
3. nonexistence of competitive equilibrium
4. existence of stable outcomes
5. proof of existence
6. properties of stable outcomes

Efficiency of stable outcomes

- outcome is in the **weak core** if no blocking coalition that can strictly improve the utilities of all members by recontracting only among themselves
  - grand coalition $\implies$ weak core outcomes are weakly Pareto efficient

Efficiency of stable outcomes

- outcome is in the **weak core** if no blocking coalition that can strictly improve the utilities of all members by recontracting only among themselves
  - grand coalition $\Longrightarrow$ weak core outcomes are weakly Pareto efficient

theorem

under net substitutes, every stable outcome is in the weak core

Efficiency of stable outcomes

- outcome is in the **weak core** if no blocking coalition that can strictly improve the utilities of all members by recontracting only among themselves
  - grand coalition $\implies$ weak core outcomes are weakly Pareto efficient

theorem

under net substitutes, every stable outcome is in the weak core

- flexible prices also critical to this result (cf. Blair (1988))

Efficiency of stable outcomes: proof strategy

- proof of efficiency goes via quasiequilibrium

proposition

under net substitutes, every stable outcome is a quasiequilibrium outcome

lemma

every quasiequilibrium outcome is in the weak core

Efficiency of stable outcomes: proof strategy

- proof of efficiency goes via quasiequilibrium

proposition

under net substitutes, every stable outcome is a quasiequilibrium outcome

lemma

every quasiequilibrium outcome is in the weak core

- just the first welfare theorem for a Hicksian economy

Efficiency of stable outcomes: proof strategy

- proof of efficiency goes via quasiequilibrium

proposition

under net substitutes, every stable outcome is a quasiequilibrium outcome

- subtle statement that relies on net substitutes (unlike converse)
- proof based on applying analogous result for TU economies (Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp, 2013) in a Hicksian economy

lemma

every quasiequilibrium outcome is in the weak core

- just the first welfare theorem for a Hicksian economy

Pairwise stability

definition

given an income profile, an outcome A is **pairwise stable** if it is individually rational and not blocked by any set consisting of a single contract

- under gross substitutes, if $\{z_1, \ldots, z_k\}$ blocks A, then so does each $\{z_\ell\}$

Pairwise stability

definition

given an income profile, an outcome A is **pairwise stable** if it is individually rational and not blocked by any set consisting of a single contract

- under gross substitutes, if $\{z_1, \ldots, z_k\}$ blocks A, then so does each $\{z_\ell\}$
- under net substitutes, there can be gross complementarities...

Pairwise stability

definition

given an income profile, an outcome A is **pairwise stable** if it is individually rational and not blocked by any set consisting of a single contract

- under gross substitutes, if $\{z_1, \ldots, z_k\}$ blocks A, then so does each $\{z_\ell\}$
- under net substitutes, there can be gross complementarities...

theorem

under net substitutability, stability and pairwise stability coincide

- so "simple" blocks suffice even in the presence of income effects
  - despite possibility of gross complementarities among large sets of contracts

Pairwise stability: role of prices

- under gross substitutes, if $\{z_1, \ldots, z_k\}$ blocks A, then so does each $\{z_\ell\}$
- in general, with fixed prices, pairwise stable outcomes can be unstable
  - even under net substitutes, example in paper

Pairwise stability: role of prices

- under gross substitutes, if $\{z_1, \ldots, z_k\}$ blocks A, then so does each $\{z_\ell\}$
- in general, with fixed prices, pairwise stable outcomes can be unstable
  - even under net substitutes, example in paper
- moving prices helps undo income effects, mitigate gross complementarities

proposition

given an income profile, let A be an individually rational outcome.
if $\{(\omega_1, p_{\omega_1}), \ldots, (\omega_k, p_{\omega_k})\}$ blocks A, then for some ℓ, p'_{ω_ℓ}, $\{(\omega_\ell, p'_{\omega_\ell})\}$ blocks A

Pairwise stability: role of prices

- under gross substitutes, if $\{z_1, \ldots, z_k\}$ blocks A, then so does each $\{z_\ell\}$
- in general, with fixed prices, pairwise stable outcomes can be unstable
  - even under net substitutes, example in paper
- moving prices helps undo income effects, mitigate gross complementarities

proposition

given an income profile, let A be an individually rational outcome.
if $\{(\omega_1, p_{\omega_1}), \ldots, (\omega_k, p_{\omega_k})\}$ blocks A, then for some ℓ, p'_{ω_ℓ}, $\{(\omega_\ell, p'_{\omega_\ell})\}$ blocks A

- proof also goes via quasiequilibrium
  - uses coincidence between solution concepts in a Hicksian economy (Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp, 2013, 2021)
- so although "simple" blocks still suffice, budget constraints and income effects can "simplifying a block" more subtle

Revisiting the structure of the set of stable outcomes

despite existence of stable outcomes, give example in the paper showing:

- set of stable outcomes may not form a lattice
  - buyer-optimal and seller-optimal stable outcomes may not exist
  - intuition: budget constraints can generate opposed interests between sellers

Revisiting the structure of the set of stable outcomes

despite existence of stable outcomes, give example in the paper showing:

- set of stable outcomes may not form a lattice
    - buyer-optimal and seller-optimal stable outcomes may not exist
    - intuition: budget constraints can generate opposed interests between sellers

- the “Lone Wolf Theorem” (or “Rural Hospitals Theorem”) can fail
    - agents can be matched in one stable outcome (and receive higher than their autarky utility) but unmatched in others
    - intuition: raising one worker’s salary can make another worker unaffordable

Revisiting the structure of the set of stable outcomes

despite existence of stable outcomes, give example in the paper showing:

- set of stable outcomes may not form a lattice
  - buyer-optimal and seller-optimal stable outcomes may not exist
  - intuition: budget constraints can generate opposed interests between sellers
- the "Lone Wolf Theorem" (or "Rural Hospitals Theorem") can fail
  - agents can be matched in one stable outcome (and receive higher than their autarky utility) but unmatched in others
  - intuition: raising one worker's salary can make another worker unaffordable
- there may be no stable matching mechanism that is strategy-proof for all unit-supply sellers (or for all unit-demand buyers)
  - intuition: misreporting a value can lower others' salaries (standard), which can make more budget available for the misreporter

Conclusion

- prices may not clear markets with indivisibilities and budget constraints
- nevertheless, we show that stable outcomes exist and are efficient even in the presence of budget constraints under the net substitutes condition

Conclusion

- prices may not clear markets with indivisibilities and budget constraints
- nevertheless, we show that stable outcomes exist and are efficient even in the presence of budget constraints under the net substitutes condition
- so stable outcomes can exist despite the presence of gross complementarities
  - and agents can even focus on "simple" blocks to identify stable outcomes
  - but gross complementarities make structure of set of stable outcomes break

Conclusion

- prices may not clear markets with indivisibilities and budget constraints
- nevertheless, we show that stable outcomes exist and are efficient even in the presence of budget constraints under the net substitutes condition
- so stable outcomes can exist despite the presence of gross complementarities
  - and agents can even focus on "simple" blocks to identify stable outcomes
  - but gross complementarities make structure of set of stable outcomes break
- price flexibility plays a critical role in our results, unlike previous work

Conclusion

- prices may not clear markets with indivisibilities and budget constraints
- nevertheless, we show that stable outcomes exist and are efficient even in the presence of budget constraints under the net substitutes condition
- so stable outcomes can exist despite the presence of gross complementarities
  - and agents can even focus on "simple" blocks to identify stable outcomes
  - but gross complementarities make structure of set of stable outcomes break
- price flexibility plays a critical role in our results, unlike previous work

implications for auction design with budget constraints:

- gross complementarities can cause problems for dynamic auctions
- but sealed-bid auctions that implement stable outcomes may work well
  - e.g., versions of the Product-Mix Auction (Klemperer, 2010; Milgrom, 2009)

Thank you!

Summary of results