Introduction to Markets for Indivisible Goods

Alex Teytelboym
Oxford

Special Lectures at the University of Tokyo

The University of Tokyo Market Design Center (UTMD) and
Center for International Research on the Japanese Economy (CIRJE)
September, 2021

European Research Council
Established by the European Commission

Outline of this lecture

- Recalling standard results from GE
- Why markets for indivisible goods?
- Results in transferable utility economies
- What about income effects?
- Roadmap for the next three lectures

General model of an exchange economy

Ingredients:

- finite set N of goods,
- finite set J of agents,
- $\mathbf{x}^j \in X^j$ are feasible consumption bundles for j,
- agent j's utility function is $U^j(\mathbf{x})$,
- agent j's endowment is $\mathbf{w}^j \in X^j$ (assume feasible),
- prices are $\mathbf{p} \in \mathbb{R}^N$,
- $D^j_{\mathrm{M}}(\mathbf{p}, \mathbf{w}^j)$ is the (Marshallian) demand correspondence, i.e., set of feasible consumption bundles that maximize agent j's utility.

definition

Given endowment $(\mathbf{w}^j)_{j \in J}$, pair $((\mathbf{x}^j)_{j \in J}, \mathbf{p})$ is a *competitive equilibrium* if $\mathbf{x}^j \in D^j_{\mathrm{M}}(\mathbf{p}, \mathbf{w}^j)$ for each $j \in J$ and markets for all goods clear.

Markets for divisible goods and general equilibrium

Suppose that for all $j \in J$:

- goods are divisible: $X^j \subseteq \mathbb{R}^N$ is convex;
- $U^j(\mathbf{x})$ is continuous, monotone, and concave;
- endowments are positive, i.e., $\mathbf{w}^j \gg 0$ for all $j \in J$.

Markets for divisible goods and general equilibrium

1. Equilibrium exists under weak conditions (Arrow and Debreu (1954); McKenzie (1954))
2. Generically, the number of equilibrium price vectors is finite (Debreu, 1970) and odd (Dierker, 1972).
3. Anything goes: aggregate demand places very restrictions on individual demands (Sonnenschein (1973), Mantel (1974), Debreu (1974)) and equilibrium entails very few restrictions on the set of equilibrium prices (Mas-Colell, 1977).
4. The core is larger than the set of competitive equilibrium allocations.
5. Tâtonnement works only under stronger conditions on preferences (Arrow and Hurwicz, 1958).
6. Computing equilibrium prices precisely is hard (Scarf, 1973; Papadimitriou, 1994).

Markets for indivisible goods

MWG (p. 598): "the most substantial [assumption for the existence of equilibrium] concerns convexity"

Many markets are thin and involve trade of highly heterogeneous goods. Indivisibilities can play a important role in...

- **exchange**: housing markets, markets for used cars...
- **auctions**: spectrum auctions, ad slots...
- **labour markets**: specialized jobs...
- **production**: highly specific inputs, machines...

Model for markets with indivisible goods

For the rest of the lecture assume the following:

- all goods except one good x_0 called "money" (the numeraire) are *in*divisible;
- set I of indivisible goods;
- $X_I^j \subseteq \mathbb{Z}^I$ of feasible bundles of indivisible goods;
- $\mathbf{x} = (x_0, \mathbf{x}_I) \in X^j$ are feasible consumption bundles.

Transferable Utility Economies

Transferable utility

- We will assume that $U^j(\mathbf{x}) = V^j(\mathbf{x}_I) + x_0$ for some *valuation* $V^j : X_I^j \to \mathbb{R}$ and money $x_0 \in \mathbb{R}$.
- Efficient outcomes are found by maximizing the sum of valuations.
- Endowments do not affect demand, so we can write demand simply as $D^j(\mathbf{p})$.

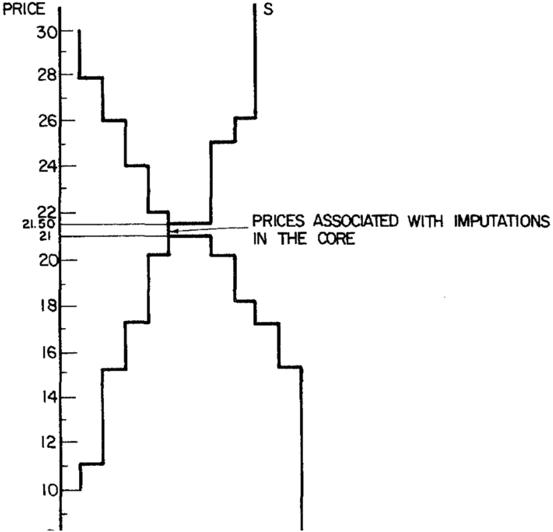
Simplest possible case

- Suppose that there is one indivisible good $|I| = 1$.
- Moreover, agents demand at most one unit.
- Perhaps you would like to buy a horse?. . .

Böhm-Bawerk's (1888) horses

Buyers.		Sellers.	
A_1 values a horse at . . (and will buy at any price under)	£30	B_1 values a horse at . . (and will sell at any price over)	£10
A_2 ,, ,,	£28	B_2 ,, ,,	£11
A_3 ,, ,,	£26	B_3 ,, ,,	£15
A_4 ,, ,,	£24	B_4 ,, ,,	£17
A_5 ,, ,,	£22	B_5 ,, ,,	£20
A_6 ,, ,,	£21	B_6 ,, ,,	£21:10s.
A_7 ,, ,,	£20	B_7 ,, ,,	£25
A_8 ,, ,,	£18	B_8 ,, ,,	£26
A_9 ,, ,,	£17		
A_{10} ,, ,,	£15		

Böhm-Bawerk (1888) horses: 5 horses traded



Schotter, IJGT, 1974

Exchange economy $\leftrightarrow$ two-sided market

- Redefine J to be "buyers" who have the same utility function as before but own nothing.
- Redefine I to be "sellers" who have a zero utility function and own goods I.

proposition (Bikhchandani and Mamer, 1997; Ma, 1998)

Competitive equilibrium exists in the exchange economy if and only if it exists in the modified two-sided market.

Assignment market

- Let's enrich the model a bit.
- Suppose now that $X_I^j = \{0, 1\}^I$, i.e., there are multiple heterogeneous goods, but only unit of each good.
- Moreover, as before, each agent owns at most one object (seller) or demands at most one object (buyer).

Assignment market

- Denote by $v_{ij} \geq 0$ the surplus created good i (owned by seller i) is bought by buyer j.
- If buyer j buys good i, his utility is $u_j^b = v_{ij}^b - p_i$.
- If seller i sells his good i, he gets $u_i^s = p_i - v_i^s$
- Denote by α_{ij} the (fractional) assignment of good i to agent j. What's the efficient assignment?

$$\max_{i,j} \quad \sum_i \sum_j v_{ij}\alpha_{ij}$$

$$\text{s.t.} \quad \sum_i \alpha_{ij} = 1 \quad \text{for all } j \in J,$$

$$\sum_j \alpha_{ij} = 1 \quad \text{for all } i \in I,$$

$$\alpha_{ij} \geq 0 \quad \text{for all } i \in I \text{ and } j \in J.$$

- This problem must have an integral solution. Why?

Assignment market

- Recall that u_i^s the utility/profit of seller i and u_j^b the utility/profit of buyer j. Now consider the dual problem:

$$\min_{i,j} \quad \sum_i u_i^s + \sum_j u_j^b$$
$$\text{s.t.} \quad u_i^s + u_j^b \geq v_{ij} \quad \text{for all } i \in I \text{ and } j \in J.$$

- Strong duality tells us that the value of the max problem is equal to the value of the min problem.
- Why do we need this? Recall that $p_i = v_i^s + u_i^s$.
- Duality gives us prices that support the efficient allocation!..

Assignment market

theorem (Koopmans and Beckmann, 1957)

There exists a competitive equilibrium in the assignment market.

Intuition for the proof: Primal gives us the allocation, dual gives us the prices.

theorem (Shapley and Shubik, 1971)

Competitive equilibrium outcomes coincide with the core.

Intuition for the proof: The constraints in the dual gives you feasibility and objective gives you non-improvability.

Assignment market

theorem (Shapley and Shubik, 1971)

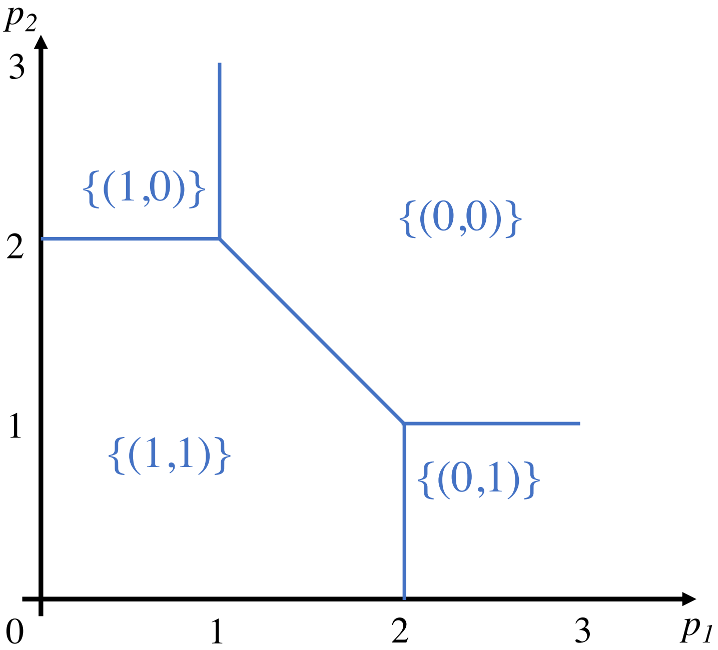
There exist minimum-price and maximum-price competitive equilibria.

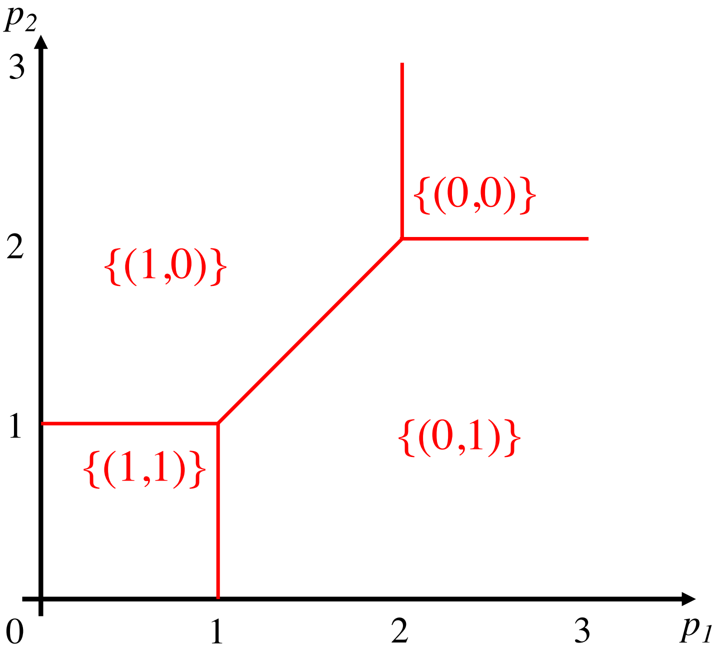
Intuition for the proof: You can add/subtract a small constant to all prices without affecting the equilibrium allocation.

Multi-good demand

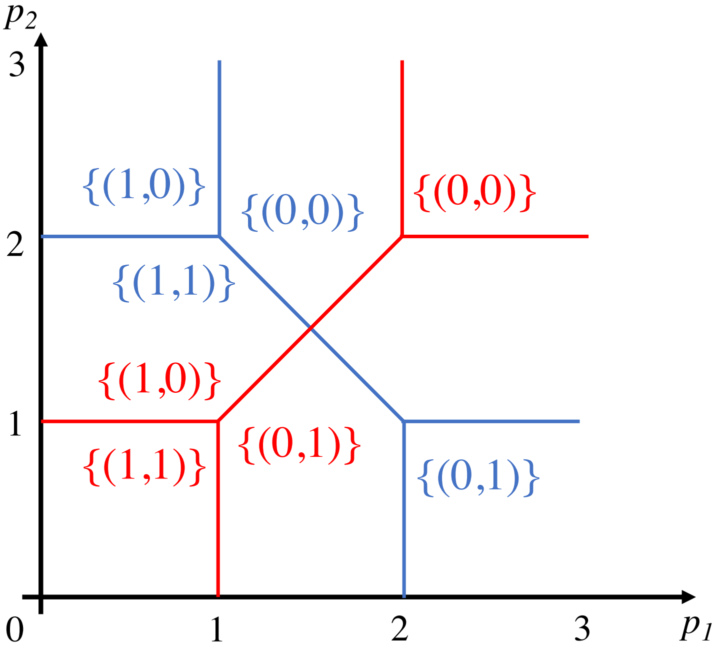
- Suppose there two goods. Seller owns both goods and values them at nothing. There are two buyers j and k with the following valuations.

$\mathbf{x}_I$	$(1,0)$	$(0,1)$	$(1,1)$
$V^j(\mathbf{x}_I)$	1	1	3
$V^k(\mathbf{x}_I)$	2	2	3


p_2
3
2
1
0
{(1,0)}
{(0,0)}
{(1,1)}
{(0,1)}
1
2
3
p_1



p_2
3
2
1
0
1
2
3
p_1
{(0,0)}
{(1,0)}
{(0,1)}
{(1,1)}



p_2
3
2
1
0
1
2
3
p_1
{(1,0)}
{(0,0)}
{(0,0)}
{(1,1)}
{(1,0)}
{(0,1)}
{(1,1)}
{(0,1)}

Competitive equilibrium with multiple goods

Denote by $\mathbf{y}_I$ the vector of total endowment of indivisible goods.

$$\text{LPRIP} = \max_{(\alpha^j)_{j\in J}} \quad \sum_{j\in J} \sum_{\mathbf{x}_I^j \in X_I^j} \alpha^j_{\mathbf{x}_I^j} V^j(\mathbf{x}_I^j)$$

$$\text{s.t.} \quad \sum_{\mathbf{x}_I^j \in X_I^j} \alpha^j_{\mathbf{x}_I^j} = 1 \quad \text{for all } j \in J,$$

$$\sum_{j\in J} \sum_{\mathbf{x}_I^j \in X_I^j} \alpha^j_{\mathbf{x}_I^j} \mathbf{x}_I^j = \mathbf{y}_I,$$

$$\alpha^j \in \mathbb{R}_{\geq 0}^{X_I^j} \alpha^j \in \{0,1\}^{X_I^j} \quad \text{for all } j \in J.$$

theorem (Bikhchandani and Mamer, 1997)

Competitive equilibrium exists if and only if the values of the optimal solutions to the IP and LPR coincide.

Competitive equilibrium and the core with multiple goods

- Consider a cooperative game in characteristic form (J, v) where v is the value function $v : 2^J \to \mathbb{R}$.
- Bondareva-Shapley Theorem tells us the core is non-empty if and only if for every function $\gamma : 2^J \setminus \{\varnothing\}$ such that

$$\sum_{S \in 2^J : j \in S} \gamma(S) = 1 \quad \text{we have that} \quad \sum_{S \in 2^J \setminus \{\varnothing\}} \gamma(S) v(S) \leq v(J).$$

- But unfortunately competitive equilibrium in the exchange economy does not coincide with core of the game if the coalitional form is generated by considering coalitions of agents...
- Solution: transform it into an two-sided market!

theorem (Ma, 1998)

Competitive equilibrium exists if and only if coalitional form game in the corresponding two-sided market is balanced.

Substitutes

- We want a more interpretable condition on preferences for equilibrium existence.

definition (Kelso and Crawford, 1982; Ausubel and Milgrom, 2002)

A valuation V^j is a *substitutes valuation* if for all price vectors $\mathbf{p}_I$ and $\lambda > 0$, whenever $D^j(\mathbf{p}) = \{\mathbf{x}_I\}$ and $D^j(\mathbf{p} + \lambda \mathbf{e}^i) = \{\mathbf{x}'_I\}$, we have that $x'_k \geq x_k$ for all goods $k \neq i$.

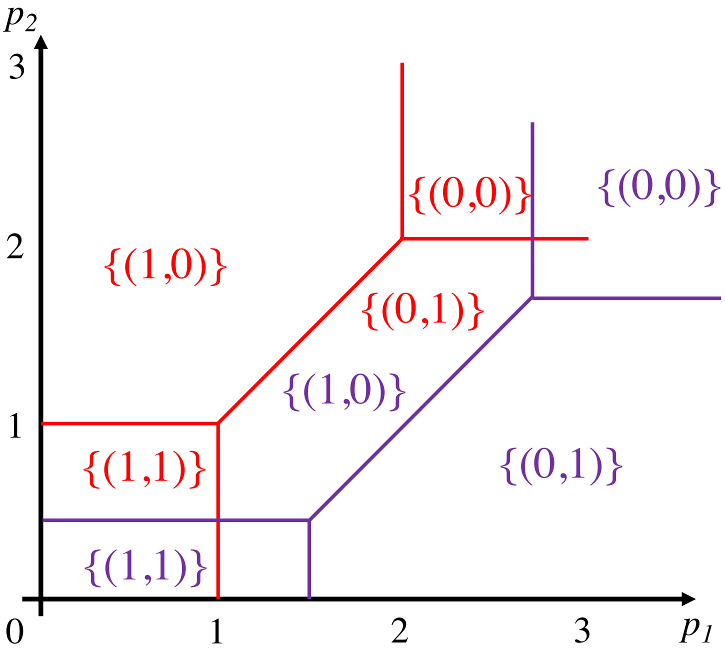
theorem (Kelso and Crawford, 1982)

If all agents have substitutes valuations, then competitive equilibria exist.

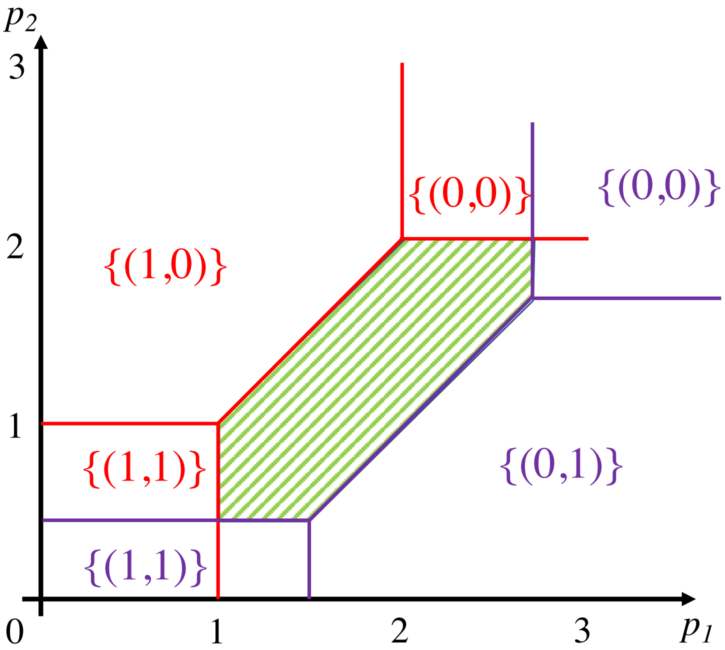
Existence under substitutes

- Discretize prices, so demand is always single-valued on the grid.[1]
- Start prices very low. If more than one firm demands the good, increase its price.
- By substitutability, as prices of some goods rise, demand for other goods weakly increases.
- Eventually, the market for each good clears.
- Take limits, obtain equilibrium in the economy with continuous prices.

[1] Not necessary but slightly fiddlier, see Gul and Stachetti (1999).



p_2
3
2
1
0
1
2
3
p_1
{(1,0)}
{(0,0)}
{(0,0)}
{(0,1)}
{(1,0)}
{(1,1)}
{(0,1)}
{(1,1)}



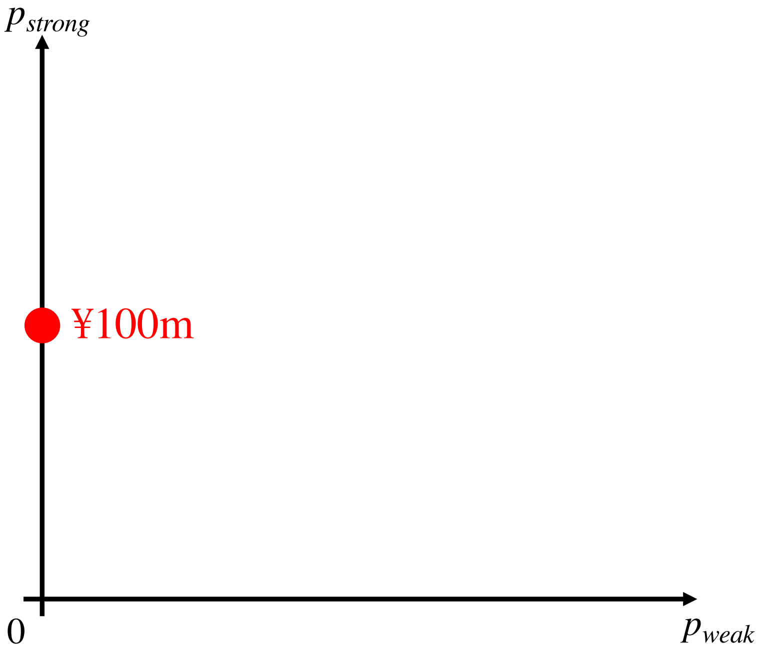
p_2
3
2
1
0
1
2
3
p_1
{(0,0)}
{(0,0)}
{(1,0)}
{(1,1)}
{(0,1)}
{(1,1)}

Facts about substitutes (Gul and Stachetti, 1999)

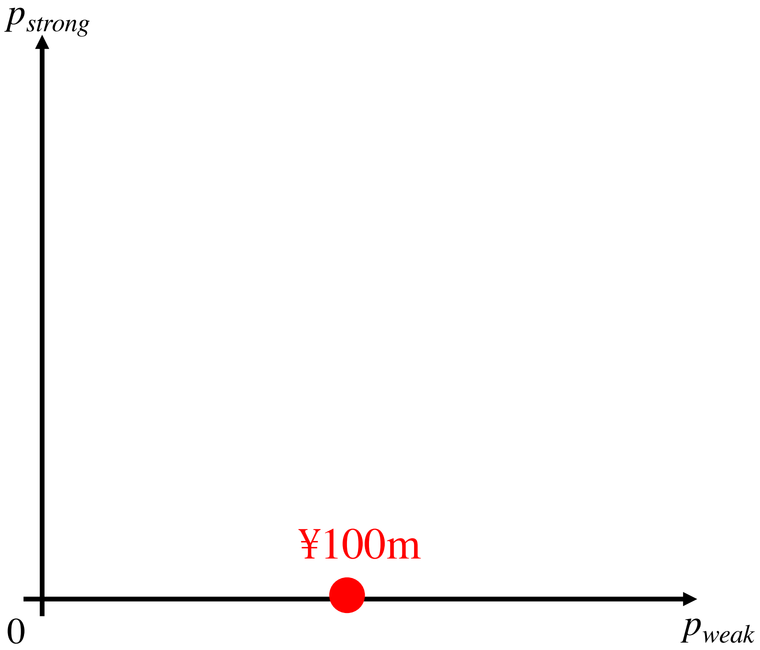
- Tight connection between monotonic auctions and Deferred Acceptance Algorithm of Gale and Shapley (1962).
- Equilibrium prices form a (complete) lattice.
- Ascending/descending auction finds the lowest/highest equilibrium prices.
- In a large enough replica economy, lowest equilibrium prices "coincide" with Vickrey-Clarke-Groves payments.
- Substitutability forms a maximal domain of preferences for existence of equilibrium.

theorem (∼Gul and Stachetti, 1999)

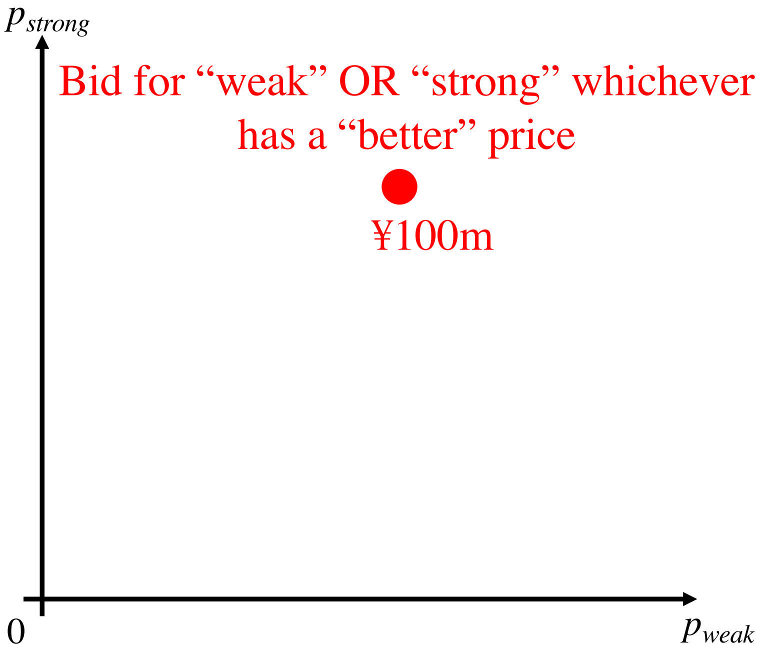
If $|J| \geq 2$, agent j demands at most one unit of each good, and V^j is not a substitutes valuation, then there exist substitutes valuations $V^j : \{0,1\}^I \to \mathbb{R}$ for agents $k \neq j$ for which no competitive equilibrium exists.

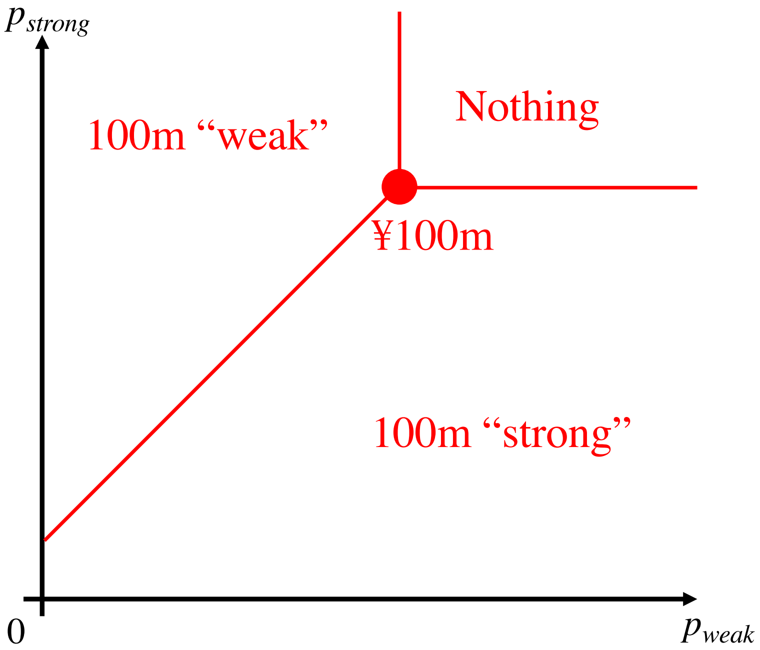


p_{strong}
¥100m
0
p_{weak}

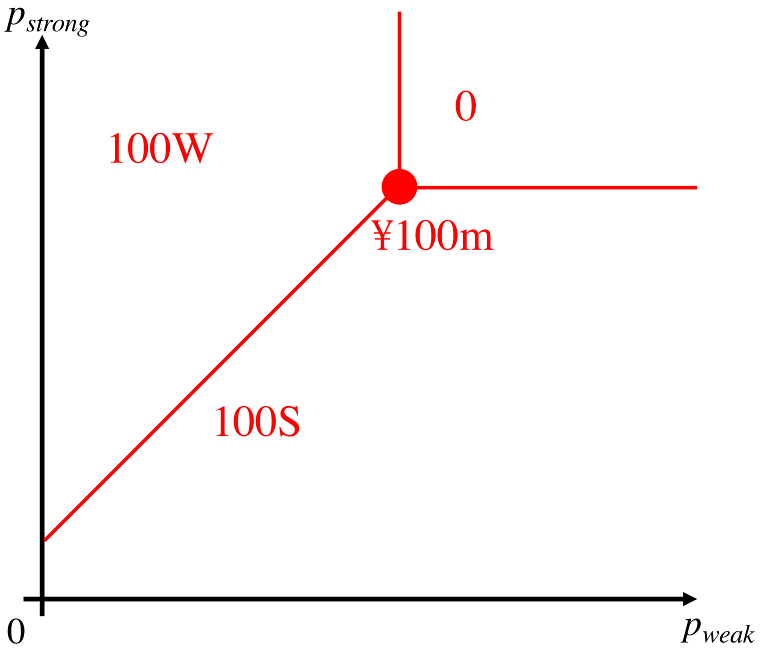


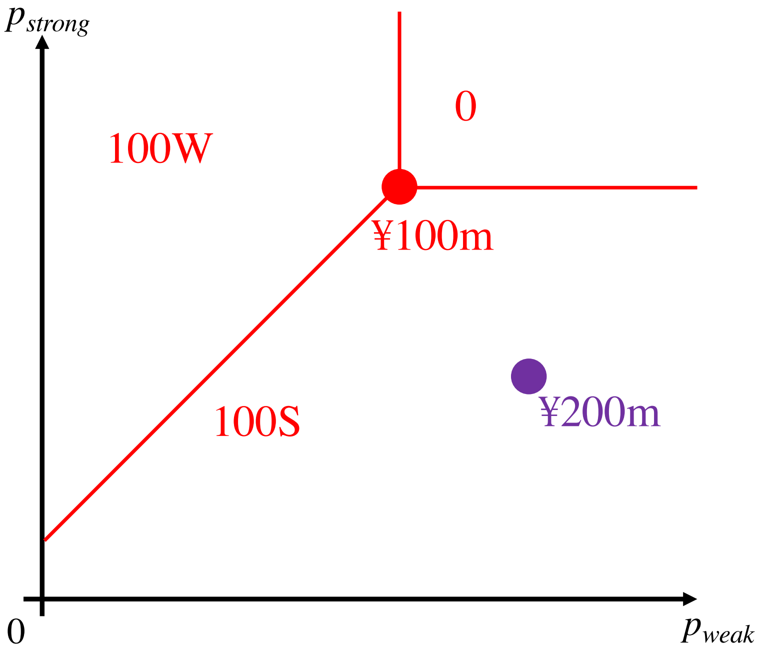
p_{strong}
¥100m
0
p_{weak}


p_{strong}
Bid for "weak" OR "strong" whichever has a "better" price
¥100m
0
p_{weak}

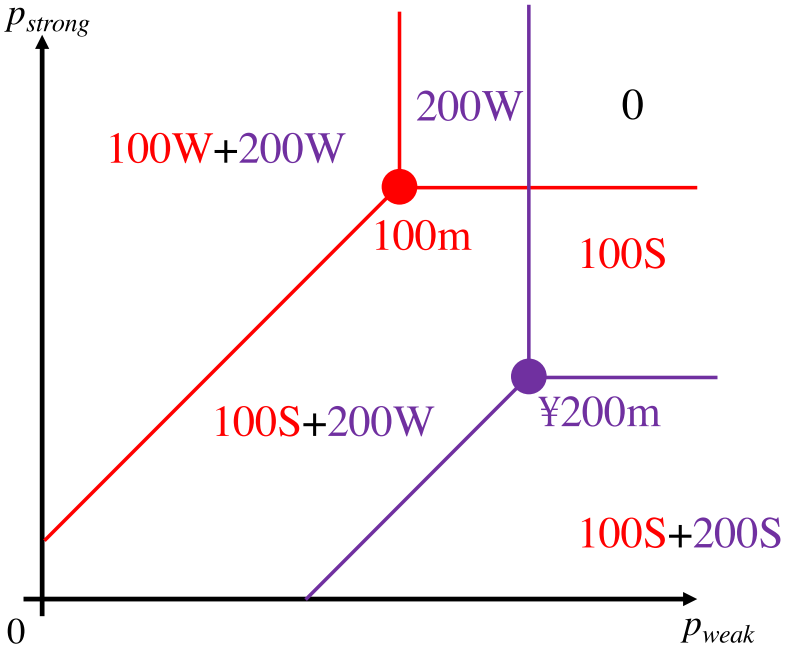


p_{strong}
100m "weak"
Nothing
¥100m
100m "strong"
0
p_{weak}


p_{strong}
0
100W
¥100m
100S
0
p_{weak}



p_{strong}
0
100W
¥100m
¥200m
100S
0
p_{weak}



p_{strong}
200W
0
100W+200W
100m
100S
100S+200W
¥200m
100S+200S
0
p_{weak}

Multiple units

- If there are multiple units of some goods, then substitutes valuations are insufficient for existence.
- Intuition: I can substitute 1 unit of a good for 2 units of another good and these two units are then complementary to each other.
- Instead, the ascending auction will terminate at a *pseudo*-equilibrium (equilibrium of the convexified economy).
- To guarantee that equilibrium exists and that Vickrey outcomes are in the core, require *strong substitutability*, i.e., that goods are substitutes *when each unit of a good is treated as a separate good* (Milgrom and Strulovici, 2009).

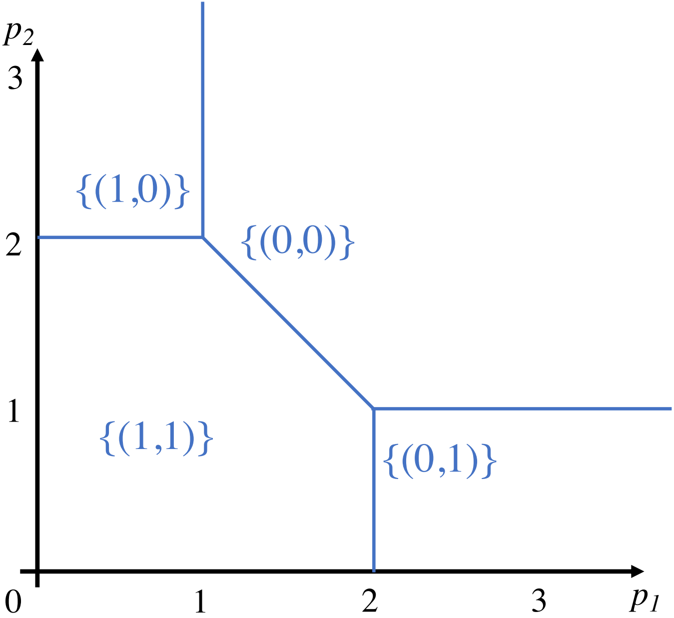
Beyond substitutes: Substitutes and complements

- Suppose we could partition goods into two sets: S_1 (tables/trousers/left shoes) and S_2 (chairs/shirts/right shoes).
- We say that agents have *substitutes and complements valuations* if all agents regard items in S_1 as substitutes, in S_2 as substitutes but any item from S_1 and an item from S_2 as complements.

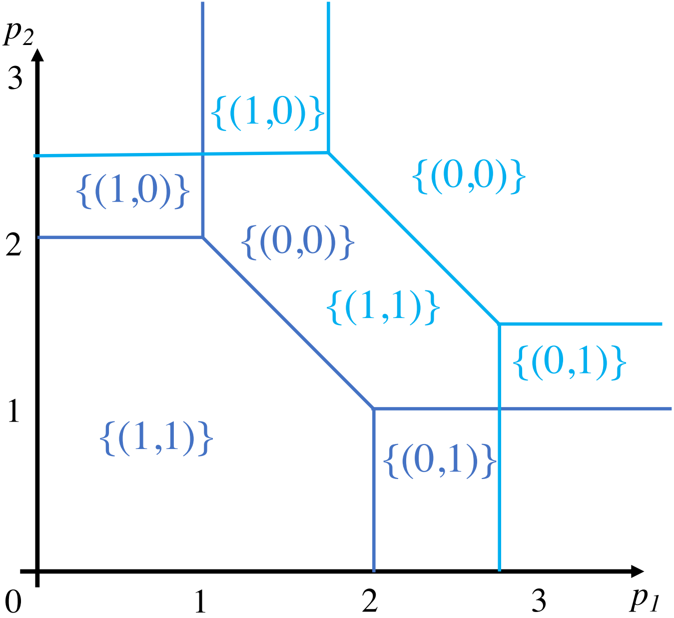
theorem (Sun and Yang, 2006)

If agents have substitutes and complements valuations, then competitive equilibria exist.

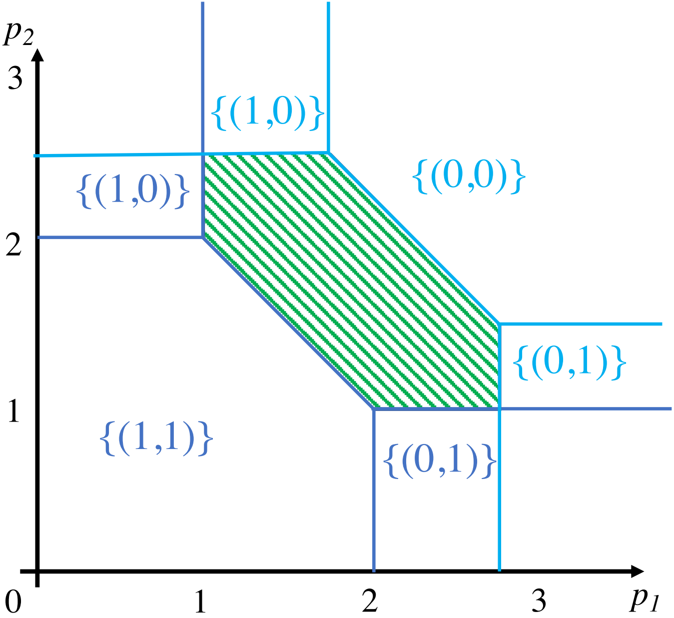
- This has been substantially generalized (Danilov et al., 2001; Baldwin and Klemperer, 2019): see Lecture 2!
- Tâtonnement process works by starting prices of goods in S_1 low and the goods in S_2 high (Sun and Yang, 2009).



p_2
3
2
1
0
1
2
3
p_1
{(1,0)}
{(0,0)}
{(1,1)}
{(0,1)}



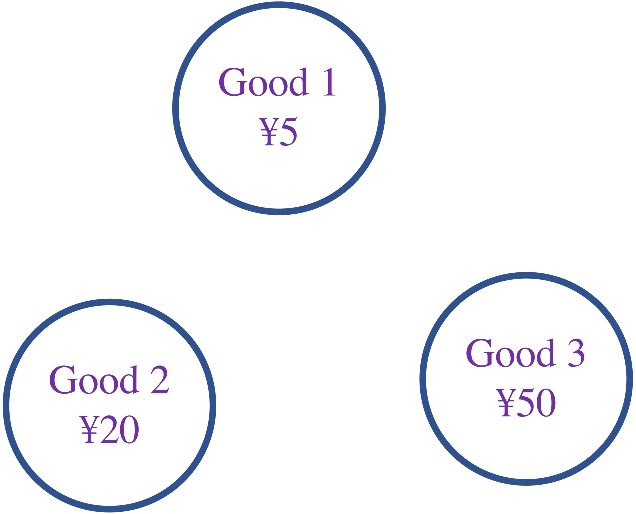
p_2
3
2
1
0
1
2
3
p_1
{(1,0)}
{(1,0)}
{(0,0)}
{(0,0)}
{(1,1)}
{(1,1)}
{(0,1)}
{(0,1)}

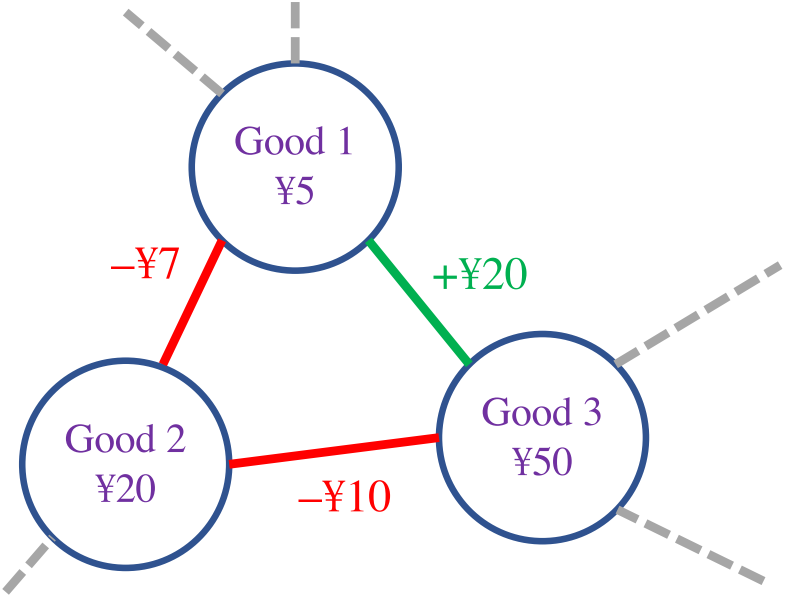


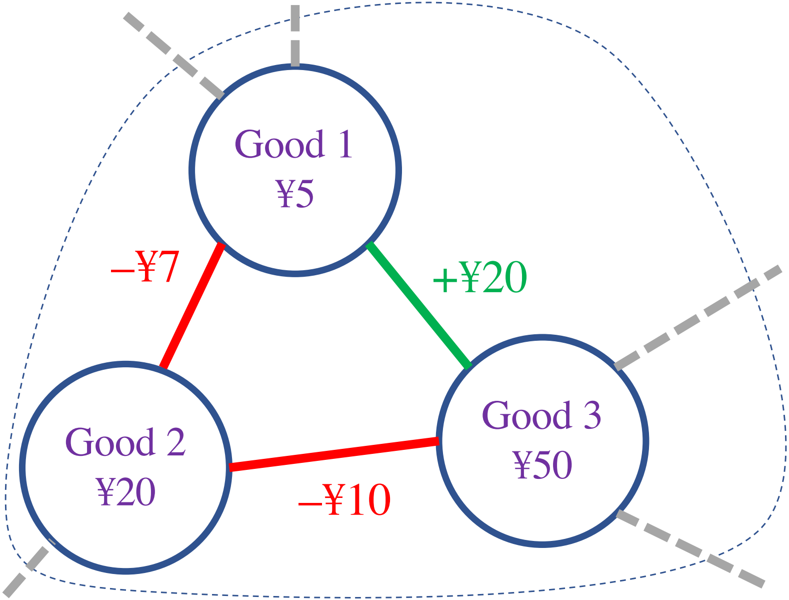
p_2
3
2
1
0
1
2
3
p_1
{(1,0)}
{(1,0)}
{(0,0)}
{(0,1)}
{(1,1)}
{(0,1)}

Beyond substitutes: graphical valuations

- Suppose that each agents' valuations over goods can be represented by a "value graph".

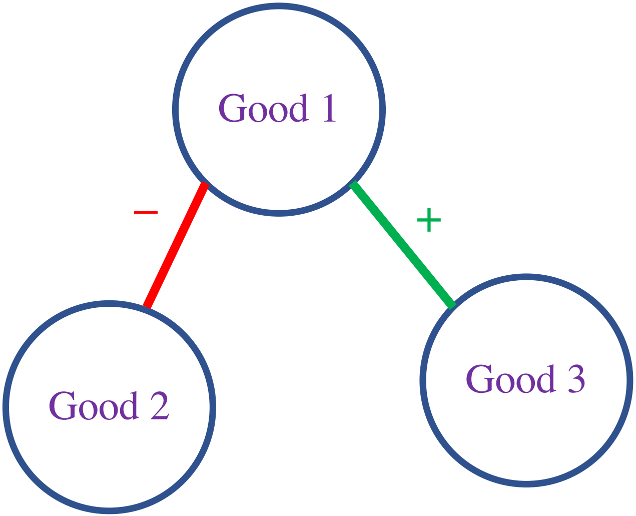

Good 1
¥5
Good 2
¥20
Good 3
¥50


Good 1
¥5
Good 2
¥20
Good 3
¥50
–¥7
+¥20
–¥10


Good 1
¥5
–¥7
+¥20
Good 2
¥20
–¥10
Good 3
¥50

Beyond substitutes: graphical valuations

- Suppose that each agents' valuations over goods can be represented by a "value graph".
- Assume, moreover, than the graph is a tree and is sign-consistent.


Good 1
−
+
Good 2
Good 3

Beyond substitutes: graphical valuations

- Suppose that each agents' valuations over goods can be represented by a "value graph".
- Assume, moreover, than the graph is a tree and is sign-consistent.

theorem (Candogan, Ozdaglar, and Parrilo, 2015)

If all agents have sign-consistent tree valuations, then competitive equilibria exist.

Income Effects

Income effects

- If a model requires that goods be indivisible, then they are "big" so income effects are natural.
- We no longer assume that utility functions are quasilinear.
- We just assume that U^j is continuous and strictly increasing in money $+$ technical assumptions (ruling out agents running out of money).
- Exchange economy is no longer isomorphic to a two-sided market...

Income effects: two-sided market

- Suppose that $X_I^j = \{0, 1\}^I$ and agents have unit demand.
- Competitive equilibrium still exists (Crawford and Knoer, 1981).
- Equilibrium prices still form a lattice (Demange and Gale, 1985).
- An allocation rule that selects the minimum-price equilibrium is strategyproof for buyers.
- In fact, it is the *only* rule that is individually rational, efficient, strategyproof, and offers no subsidy for losers (Serizawa and Morimoto, 2015).

Income effects: two-sided market

- If agents' demand multiple goods, we need to restrict preferences: e.g., Kaneko and Yamamoto (1986), van der Laan et al. (1997, 2002), and Yang (2000) assume separable preferences.

definition (Kelso and Crawford, 1982; Fleiner et al., 2019)

A utility function U^j is a *gross substitutes utility function at endowment* $\mathbf{w}^j$ if for all price vectors $\mathbf{p}_I$, and $\lambda > 0$, whenever $D^j_{\mathrm{M}}(\mathbf{p}, \mathbf{w}^j) = \{\mathbf{x}_I\}$ and $D^j_{\mathrm{M}}(\mathbf{p} + \lambda \mathbf{e}^i, \mathbf{w}^j) = \{\mathbf{x}'_I\}$, we have that $x'_k \geq x_k$ for all goods $k \neq i$.

theorem (Fleiner et al., 2019)

If all agents have gross substitutes utility functions at their endowments, then competitive equilibria exist.

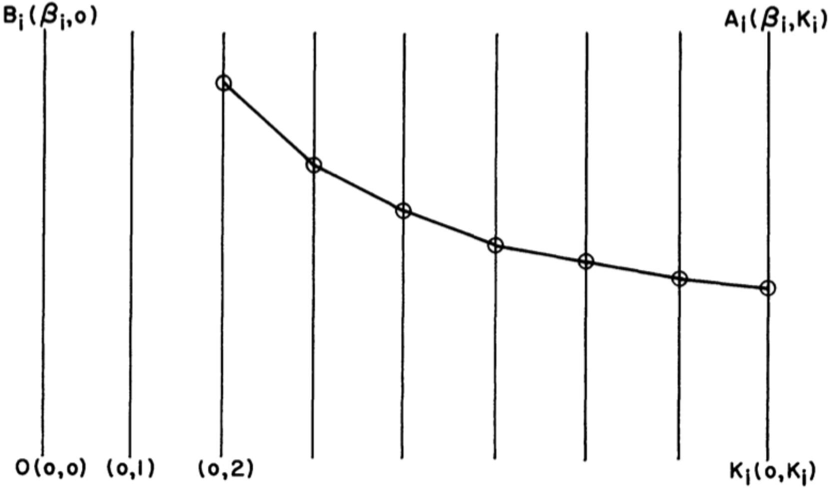
- The proof strategy is very similar to Kelso and Crawford's; most structural results go through (Schlegel, 2021).

Income effects: trouble with gross substitutes

- But does gross substitutability make sense with income effects?
- Consider a buyer. If the price of good i increases, demand for good k also *increases*.
- But when the price of good i increases, the buyer is poorer, so if good k is normal, then he *decreases* demand for good k...
- This suggests that the gross substitutability requires that goods be inferior for the buyer!

Income effects: exchange economy

- Suppose that $|I| = 1$. Henry (1970) showed that equilibrium exists...



Income effects: exchange economy

- Suppose again that $X_I^j = \{0,1\}^I$ and agents have unit demand. This is the "housing market" with income effects.
- Gross substitutability is *not* satisfied: if a price of the house that the agent owns goes up, she may switch from demanding a mediocre house to demanding a fancy house.
- But, surprisingly, equilibrium always exists!
- Proofs all use steps with topological arguments, e.g., Gale (1984) uses Knaster-Kuratowski-Mazurkewicz Lemma; Svensson (1984) uses Kakutani Fixed Point Theorem; Quinzii (1984) uses Scarf's (1967) Lemma.
- In Lecture 3, I will give very general, economically interpretable conditions for existence of equilibrium in the presence of income effects while allowing for multi-good demand.

Markets for indivisible goods: TU

1. Equilibrium requires strong assumptions on preferences (e.g., substitutes).
2. Typically, a continuum of equilibrium prices.
3. Under substitutes, lattice structure of equilibrium prices.
4. The core coincides with competitive equilibrium allocations.
5. Under substitutes, tâtonnement works.
6. Computing equilibrium prices is easy.

Roadmap for these lectures

1. This Lecture: introduction and overview
2. (Elizabeth Baldwin): preferences beyond substitutes, "demand types", Unimodularity Theorem, product-mix auctions.
3. (Alex Teytelboym) Income effects in exchange economies: substitution effects, Hicksian Demand, Equilibrium Existence Duality, net substitutes.
4. (Ravi Jagadeesan) Two-sided markets and budget constraints: stability without competitive equilibrium.

Markets for indivisible goods: Income effects vs. TU

1. Equilibrium requires strong assumptions on preferences.
2. Typically, a continuum of equilibrium prices.
3. Equilibrium prices lack structure.
4. The core coincides with competitive equilibrium allocations. With budget constraints, stable outcomes exist, but equilibrium prices might not.
5. Tâtonnement does not work.
6. Computing equilibrium prices is hard.

Thank you!