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Abstract

In this paper, we study *incomplete preferences with optimism and pessimism* (IPOP) over Anscombe-Aumann acts, a class of preference orders that may fail axioms that require the decision-maker (DM) to think contingently. The main result axiomatizes a preference order $\succsim$ represented by the following rule:

$$f \succsim g \Longleftrightarrow \min_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \max_{\mu \in C^\sharp} \int (u \circ g) d\mu,$$

for any distinct acts f and g. Here u is a utility function over outcomes, and $C^\sharp$ and $C^\flat$ are non-disjoint sets of beliefs over states of the world. This representation can be interpreted as capturing the DM's conservative attitudes toward uncertainty: An act f is deemed superior to another act g if the pessimistic expected utility of f is greater than the optimistic expected utility of g. The representation reduces to a standard SEU preference when belief sets are minimal. Conversely, when belief sets are maximal, the representation encapsulates obvious dominance, the decision rule introduced by Li (2017).

Keywords: Incomplete preferences; Anscombe-Aumann axioms; Multiple-prior models; Obvious dominance; Failures in contingent reasoning.

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1 Introduction

The dominance principle is regarded as one of the most fundamental rationality in many economic decisions under uncertainty. Experiments conducted in various environments have demonstrated, however, that subjects often do not take their dominant strategies.[1] One plausible explanation for this anomaly is as follows: Subjects have too little information to form a single scenario of how their opponents will behave, and thus, they instead consider a set of scenarios as possible. As such, they may still be able to find a dominant strategy if they can perform *contingent reasoning*, i.e., compare the consequences of different strategies conditional on each possible scenario and properly synthesize those conditional relations to develop (unconditional) dominance relations.[2] Without skills to perform contingent reasoning, however, different scenarios may be applied to evaluate different strategies, which may hamper subjects to end up with a dominant strategy. For instance, in a second-price auction, overbidding would be "rationalized" against truth-telling if subject make an *as if* assumption that the more aggressively she bids, her opponents are overwhelmed to bid lower values.

The idea of failures in contingent reasoning is not new. In philosophy, Nozick (1969) proposed strategy-dependent reasoning as a contrast to contingent reasoning to explain Newcomb's paradox. More recently, Esponda and Vespa (2019) designed laboratory experiments to detect the cause of anomalies and deduced that most anomalies can be explained by failures in contingent reasoning. In parallel with the empirical literature, Li (2017) proposes a novel solution concept known as *obvious strategy-proofness*, which assumes a special form of strategy-dependent reasoning, i.e., the consequences of an equilibrium strategy are evaluated based on its worst-case scenario, whereas the consequences of possible deviations are evaluated based on their best-case scenarios.

Our focus in this paper is on a certain form of strategy-dependent reasoning that emerge naturally by weakening Anscombe and Aumann's (1963) axioms so as not to require the decision-maker (DM) to be sophisticated in contingent reasoning. Specifically, we study a preference order $\succsim$ over Anscombe-Aumann acts represented as follows:

$$f \succsim g \Longleftrightarrow \left[\min_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \max_{\mu \in C^\sharp} \int (u \circ g) d\mu \ \text{ or } \ f = g \right].$$

Here u is the utility function over outcomes, and $C^\sharp$ and $C^\flat$ are sets of beliefs over the state space interpreted as probabilistic scenarios. This representation can be interpreted as capturing the DM's conservative attitudes toward uncertainty. Adopting this interpretation, we shall refer to this class as *incomplete preferences with optimism and pessimism* (IPOP). Since different beliefs may be active to evaluate different strategies, our representation accounts for violation of the dominance principle, or the monotonicity axiom in Anscombe and Aumann (1963).

[1] To list a few, these environments include voting (Esponda and Vespa, 2014; 2019), auctions (Kagel et al. 1987; Kagel and Levin, 1993; Charness and Levin, 2009; Li, 2017), and matching problems (Rees-Jones, 2017).

[2] The notion of contingent reasoning is originally introduced by Savage (1961) in decision theory, and the rationality for any choice to follow it is called the sure thing principle after him.

The IPOP representation continuously fills the gap between two important decision rules. As such, IPOP reduces to subjective expected utility (SEU) when $C^\sharp$ and $C^\flat$ are the same singleton. On the other hand, IPOP captures obvious dominance (Li, 2017) when $C^\sharp$ and $C^\flat$ comprise all beliefs over the state space. In general, our DM is too conservative to have a unique belief but not so conservative as to apply Li's decision rule with respect to all degenerate beliefs. This flexibility allows us to make sharper predictions about the DM's behavior than those derived from Li's decision rule, while accounting for experimental observations of deviations from the dominance principle. For instance, in a second-price auction, all bidding strategies are undominated with respect to Li's decision rule, but IPOP can yield a reasonable but not trivial range of undominated bidding strategies, i.e., we can depart from *anything goes*.

In Section 3, we present our main result (Theorem 1), which provides the axiomatic foundation of IPOP. As such, when stating our axioms, we employ two hypotheses, which will reveal some important properties of IPOP pertaining to the special treatment of constant acts. First, we assume that the Anscombe-Aumann's core axioms are maintained, at least when a constant act is involved in a choice, e.g., we adopt *C-completeness* from Bewley (2002) and *C-independence* from Gilboa and Schmeidler (1989). This hypothesis is motivated by the idea that certain prospects can be understood without performing contingent reasoning. The monotonicity axiom is also weakened in light of this idea. Second, and more importantly, we assume that a reference to certain prospects is needed whenever the DM infers rankings among general acts. Specifically, our key axiom, *C-calibration*, postulates that whenever the ranking $f \succsim g$ between distinct acts is confirmed, there must exist some constant act x that has *calibrated* this ranking, i.e., $f \succsim x$ and $x \succsim g$. In other words, transitivity mediated by constant acts is the only inference rule that can be used to make a choice between uncertain prospects whose outcomes are highly contingent.

In Section 4, we study the comparative statics of different belief sets. Similar to the case with other multiple-prior models, the DM's perceptions of uncertainty are reflected in a belief set size, i.e., the larger the belief sets $C^\sharp$ and $C^\flat$, the less complete the preference order will be. As mentioned before, the two extreme cases correspond to SEU and obvious dominance. Unlike the case of existing models, however, $C^\sharp$ and $C^\flat$ do not generally need to coincide, and thus, that the DM may perceive different amounts of uncertainty depending on whether she is optimistic or pessimistic. We introduce two extra axioms that will characterize the relative size of $C^\sharp$ and $C^\flat$ in terms of set-inclusion. These axioms jointly characterize the case of when $C^\sharp$ and $C^\flat$ are equalized.

In Section 5, we discover some interesting relations with decision rules other than SEU and obvious dominance. We first note that the intersection of IPOP and maximin expected utility (MEU) a lá Gilboa and Schmeidler (1989), as well as the intersection of IPOP and the unanimity rule a lá Bewley (2002), consists solely of SEU. This is because IPOP satisfies the standard monotonicity axiom only when $C^\sharp$ and $C^\flat$ are the same singleton. On the other hand, IPOP is less complete than the other two classes, and thus, MEU and the unanimity rule can be obtained as the extensions of IPOP. This means that an analyst can understand the revealed choice consistent with either MEU or the unanimity rule as having emerged from some underlying IPOP representation. Furthermore, we provide the full characterization of the set of all complete extensions that can be obtained from IPOP.

All omitted proofs are relegated to appendices. We prove the main theorem in Appendices A and B, and the rest of propositions in Appendix C. Lastly, we report a few more omitted results in the supplementary material of this paper.

1.1 Related Literature

Let us briefly introduce a few recent attempts in decision-theory that depart from the dominance principle. We first note that the same class of preference orders considered in this paper is being simultaneously and independently studied by Echenique et al. (2020), under the name of *twofold conservatism*. They interpret the representation as capturing the conservative principle under uncertainty, which suggests the DM to choose a new option over the status-quo only if there is a good reason to do so. Up to small differences, the axiomatizations in both papers coincide with one another.

Ellis and Piccione (2017) are motivated by anomalies in financial markets wherein investors reveal a strict preference between different portfolios with same returns. They enrich the standard Anscombe-Aumann model by adding some structures. Specifically, their framework distinguish from the left and right sides of the mixture expression $h = \alpha f + (1-\alpha)g$ even though the objective returns are the same. As such, h is interpreted as buying a single portfolio, whereas $\alpha f + (1-\alpha)g$ is interpreted as buying α units of asset f and $(1-\alpha)$ units of asset g, and their *basic correlation representations* allow the DM to strictly prefer h to $\alpha f + (1-\alpha)g$. Intuitively, this can result from wrong perceptions of the correlation between f and g. Although there are no formal relations, our paper and Ellis and Piccione (2017) are complementary at the conceptual level, as the same phenomenon can be explained from various perspectives. As such, they attribute the source of anomalies to correlation misperception, whereas our approach attributes it to failures in contingent reasoning.

Similar to our approach, some papers are built on the idea that alternatives with high contingencies may be difficult for the DM to evaluate. A few examples include Puri (2020), Saponara (2020), and Valenzuela-Stookey (2020). These papers incorporate the cognitive costs of contemplating complex objects into traditional decision models. Puri (2020) adopts the vNM framework to study preferences for simplicity, where the expected value of each lottery is subtracted by a cost that is increasing in the lottery's support size. As such, her *simplicity representations* predict violation of first order stochastic dominance, which is analogous to violation of monotonicity in this paper. We emphasize that her main focus is choice under risk where objective distributions are available, whereas our main focus is choice under uncertainty that involves the state space and its subjective quantification. Hence, this paper and Puri (2020) have quite different empirical content and potential areas of application.

Saponara (2020) and Valenzuela-Stookey (2020) adopt the Anscombe-Aumann framework as similar to this paper. They model cognitive costs as partitions of the state space that place constraints on the set of contingencies that the DM can consider. There are many formal differences between this paper and their works. For instance, *revealed reasoning representations* by Saponara (2020) maintain both completeness and monotonicity, and *simple bound representations* by Valenzuela-Stookey (2020) maintain monotonicity, whereas we weaken both axioms to account for failures in contingent reasoning.

Another difference from the aforementioned papers is that our representation utilizes no building blocks beyond those that have already appeared in multiple-prior models. This feature is particularly useful when we study formal relations with the existing models, such as those of Gilboa and Schmeidler (1989), Bewley (2002), and Ghirardato et al. (2004). Specifically, preference orders considered in these papers can be understood as the reveled choice of the DM who follows our behavioral axioms. Our analysis on this point can be understood in parallel with the work of Gilboa et al. (2010) and its generalization by Frick et al. (2020), which build a bridge between certain classes of incomplete and complete preferences. As similar to that the unanimity rule is related to MEU in Gilboa et al. (2010), and to α-MEU in Frick et al. (2020), we observe that IPOP is related to what we call *generalized α-MEU* when its compatible extensions are considered.

An important feature of IPOP is the "dual-self" perspective, which is a certain kind of strategy-dependent reasoning a lá Nozick (1969). This perspective is found in some other contexts as well. One example is *interval orders*, which date back to Luce (1956) and Fishburn (1970). As formalized in our preliminary result (Lemma 1), a part of axioms considered in this paper characterizes interval orders, and thus, combines to form a novel behavioral foundation for interval orders. In the context of ambiguity, Chandrasekher et al. (2020) introduce *dual-expected utility*, in which the DM's optimism and pessimism interact differently from IPOP. Zhang and Levin (2017) provide an alternative characterization of obvious dominance (Li, 2017), but their approach is somewhat indirect compared with our results, as they maintain the completeness axiom. This point is formally discussed in Section 5.1, where we propose an additional axiom that pins down obvious dominance within IPOP.

2 The Setting

2.1 Preliminaries

Let X be a convex compact subset of a certain topological vector space, whose element $x \in X$ is called an *outcome*. There is a state space (Ω, Σ), where Ω is a compact metric space, and Σ is a field on Ω. Denote by $\Delta(\Omega, \Sigma)$, or simply $\Delta(\Omega)$, the set of all finitely additive probability measure (a.k.a. probability charges) on (Ω, Σ), and we endow it with the weak-* topology.[3] An *act* $f : \Omega \to X$ is a Σ-measurable function that maps each state ω to some outcome x. Then, the set of acts $\mathcal{F} = \{f \in X^\Omega : f \text{ is } \Sigma\text{-measurable}\}$ becomes an affine space if we define addition and scalar multiplication in a state-wise way, i.e., given acts $f, g \in \mathcal{F}$ and a number $\alpha \in (0, 1)$, the mixture $\alpha f + (1-\alpha)g$ is defined to be the act that carries an outcome $\alpha f(\omega) + (1-\alpha)g(\omega)$ in each state

[3] As is well-known, $\Delta(\Omega)$ turns out to be a compact metrizable space (and thus, separable as well), provided that Ω is a compact metric space. In particular, every closed subset of $\Delta(\Omega)$ becomes compact. For reference, see Chapter 15 of Aliprantis and Border (2006).

ω. A *constant act* is the one that takes the same outcome in every state. Denote by $\mathcal{F}^c$ the set of all constant acts. Clearly, $\mathcal{F}^c$ is isomorphic to X, so we often identify them. Slightly abusing the notation, we denote by x to express a constant act f such that $f(\omega) = x$ for all $\omega \in \Omega$.

The primitive of our analysis is a weak preference order $\succsim$ over $\mathcal{F}$. As usual, we denote by $\succ$ and $\sim$ to express the asymmetric and symmetric parts of $\succsim$, respectively. That is to say, $f \succ g$ if and only if $f \succsim g$ and $g \not\succsim f$, and $f \sim g$ if and only if $f \succsim g$ and $g \succsim f$. In general, $\succsim$ is not required to be complete, so that there may exist $f, g \in \mathcal{F}$ such that neither $f \succsim g$ nor $g \succsim f$. In that case, we say f and g are *incomparable*. Otherwise, we say f and g are *comparable*.

2.2 Representation

The following is the notion of representation that we are going to derive.

Definition 1. *A preference order $\succsim$ admits an* IPOP representation *if there exists a tuple $(u, C^\sharp, C^\flat)$, where $u : X \to \mathbb{R}$ is a continuous affine function, and $C^\sharp$ and $C^\flat$ are non-disjoint, closed, and convex subsets of $\Delta(\Omega)$, such that for all $f, g \in \mathcal{F}$,*

$$f \succsim g \iff \left[\min_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \max_{\mu \in C^\sharp} \int (u \circ g) d\mu \quad \textit{or} \quad f = g\right]. \tag{1}$$

The above representation captures two different identities of the DM, which we refer to as optimism and pessimism, and the term IPOP originates from this interpretation. An IPOP representation becomes the SEU (Anscombe and Aumann, 1963) when $C^\sharp$ and $C^\flat$ are the same singleton. On the other hand, it embodies obvious dominance (Li, 2017) when $C^\sharp$ and $C^\flat$ are equal to $\Delta(\Omega)$, in which case (1) reduces to

$$f \succsim g \iff \left[\min_{\omega \in \Omega} u(f(\omega)) \geq \max_{\omega \in \Omega} u(g(\omega)) \quad \text{or} \quad f = g\right]. \tag{2}$$

A couple of assumptions are made about belief sets $C^\sharp$ and $C^\flat$. As such, closedness and convexity are just for the sake of uniqueness results. In particular, closedness implies compactness in the current setting, so that the minimum and maximum in (1) are indeed attainable. The essential assumption in the representation is that $C^\sharp$ and $C^\flat$ have a nonempty intersection. It turns out that this condition ensures the transitivity of a preference order $\succsim$, but it has no role beyond that.[4]

Contour sets may be useful in grasping the nature of IPOP representations. Figure 1 shows the upper and lower contour sets of several $\succsim$ that admit IPOP representations. There are two states, ω_1 and ω_2, and each axis represents the corresponding coordinate of utility acts $u \circ f \in \mathbb{R}^2$. The reference points are displayed as black bullets, and the upper and lower contour sets are painted in dark green and right green, respectively. Note that the boundaries are included in contour sets, so that the green regions are closed convex subsets of $\mathbb{R}^2$. Thus, the remaining yellow regions become

[4] We prove this claim formally in Appendix D.1.

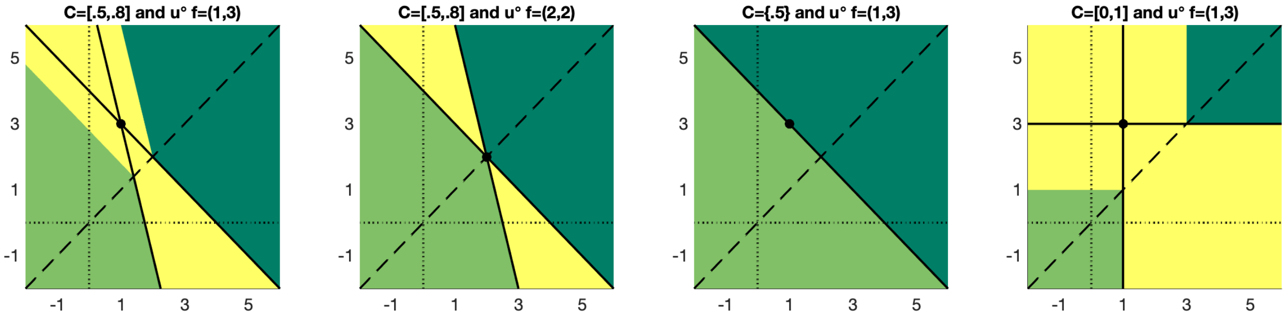

Figure 1: Upper contour sets (dark green), lower contour sets (light green), and incomparable sets (yellow) are displayed for weak preference orders $\succsim$. The reference points are displayed as black bullets. All the boundaries are included in the green regions.

open sets of incomparable utility acts. Here, we focus on the symmetric case $C \equiv C^\sharp = C^\flat$, so the contour sets are located symmetrically around the reference points.

In general, when the reference point is not on the 45° line (leftmost image), a thick incomparable region appears due to the lack of monotonicity. On the other hand, when the reference point is on the 45° line (middle-left image), the thickness disappears around the reference point, indicating that the contour sets coincide with those generated by the Bewley preference. Importantly, the thickness depends on how asymmetric the reference point is. As such, the incomparable region expands as the reference point moves away from the 45° line. As to the effect of different belief sets, SEU corresponds to the special case when C is a singleton (middle-right image). On the other hand, obvious dominance emerges when C contains all beliefs on the binary state space (rightmost image).

3 Axiomatization Results

3.1 Axioms

Let us present the axioms we use to characterize IPOP representations. The first axiom is a collection of standard assumptions.

A1 (Structural Assumptions)**.** $\succsim$ is a non-degenerate preorder such that for any $f \in \mathcal{F}$ there exist some $x, y \in \mathcal{F}^c$ for which $x \succsim f$ and $f \succsim y$.

In what follows, we weaken the remaining Anscombe-Aumann's axioms, so they are maintained by our bounded rational DM. To this end, we shall make the following two premises. First, we assume that our DM can determine the value of a constant act more easily than determining the value of a general act. Second, our DM uses constant acts as references for making a choice between general acts. In light of these premises, we shall assume that our primitive order $\succsim$ satisfies the standard axioms, at least when a constant act is involved in a comparison.

The second and third axioms are adopted from Bewley's (2002) representation theorem of the Knightian uncertainty model. The fourth axiom is adopted from Gilboa and Schmeidler's (1989) representation theorem of MEU preferences.

A2 (C-Completeness)**.** For any $x, y \in \mathcal{F}^c$, either $x \succsim y$ or $y \succsim x$ holds.

A3 (C-Continuity)**.** For any $f \in \mathcal{F}$, $\{x \in \mathcal{F}^c : x \succsim f\}$ and $\{y \in \mathcal{F}^c : f \succsim y\}$ are closed in $\mathcal{F}^c$.

A4 (C-Independence)**.** For any $f, g \in \mathcal{F}$ and $x \in \mathcal{F}^c$, we have $f \succsim g$ if and only if $\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x$ for all $\alpha \in (0, 1)$.

The next axiom is another weakening of independence, which postulates the convexity of the upper and lower contour sets of a constant act.[5]

A5 (Secure-Potential Independence)**.** For any $f, g \in \mathcal{F}$ and $x \in \mathcal{F}^c$, if $f \succsim x$ and $g \succsim x$, then $\frac{1}{2}f + \frac{1}{2}g \succsim x$. Also, if $x \succsim f$ and $x \succsim g$, then $x \succsim \frac{1}{2}f + \frac{1}{2}g$.

To interpret this axiom, we refer to the notions of the "security" and "potential" of acts, introduced by Kopylov (2009), which capture two different ways of measuring the quality of a general act by comparing it with certain prospects.[6] The axiom states that the DM can understand the quality of fairly mixed acts better than the quality of each original act, measured in either way. That is, the first half of the axiom states that if both f and g are understood to be at least as secure as x, then the fair mixing of f and g should also be understood to be as secure as x. Similarly, the second half of the axiom states that the potential of their fair mixing should be more easily understood than the potential of each original one. As such, we are taking as a working hypothesis that the DM has a better understanding of the quality of fairly mixed acts than the quality of each original act, since the mixture will get closer to a certain prospect than the original acts.

The next two axioms weaken the standard monotonicity condition.

A6 (C-Monotonicity)**.** For any $f \in \mathcal{F}$ and $x \in \mathcal{F}^c$, if $f(\omega) \succsim x$ for all $\omega \in \Omega$, then $f \succsim x$. Also, if $x \succsim f(\omega)$ for all $\omega \in \Omega$, then $x \succsim f$.

A7 (Secure-Potential Equivalence)**.** For any $f, g \in \mathcal{F}$ and $x \in \mathcal{F}^c$, if $f(\omega) \sim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim x$ if and only if $g \succsim x$, and $x \succsim f$ if and only if $x \succsim g$.

[5] The first half of **A5** has essentially the same role as *uncertainty aversion* in Gilboa and Schmeidler (1989), i.e., it makes an upper contour set convex. On the other hand, the second half assures the convexity of a lower contour set, which is less common in the literature. As such, in the current context, we do not take this axiom as the DM's attitude toward uncertainty, but interpret it in light of security and potential as explained below. Note that it is without loss of generality to assume that the mixing ration is a half. This special form implies a more general form in the presence of our continuity axiom.

[6] According to his definitions, an act f is *more secure* than g if for any constant act x, $g \succsim x$ implies that $f \succsim x$. Also, f has *more potential* than g if for any constant act x, $x \not\succsim g$ implies that $x \not\succsim f$, or equivalently, $x \succsim f$ implies that $x \succsim g$.

The axiom **A6** is a natural weakening of monotonicity, which is in effect only when at least one constant act is involved in a comparison. The axiom **A7** is a weak regularity condition saying that two acts are treated similarly in terms of security and potential whenever they are equivalent in terms of payoff realizations.[7]

The last axiom is important both conceptually and analytically. Conceptually, the axioms introduced so far have only *implicitly* postulated the bounded rationality of the DM, as they posit only what the DM can *at least do*. On the other hand, **A8** *explicitly* prescribes how the DM makes a choice without having the full ability of contingent reasoning. Analytically, **A8** has a key role in generating IPOP's dual-self perspective, which is an important feature of our decision model.

A8 (C-Calibration)**.** For any distinct acts $f, g \in \mathcal{F}$ such that $f \succsim g$, there exists a constant act $x \in \mathcal{F}^c$ for which $f \succsim x$ and $x \succsim g$.

This axiom puts constraints on the DM's ability to make a choice from general acts. It states that whenever the DM can rank distinct f and g, there must exist a constant act x that lies between them. In other words, transitivity relations mediated by constant acts are the only inference rules the DM can use to rank distinct general acts.

Meanwhile, in the presence of the other axioms, **A8** is equivalent to the following alternative axiom, which admits a different interpretation. Perhaps constant acts are those with values that are easiest to evaluate, therefore, if neither f nor g is comparable with a constant act, such a pair must not be comparable themselves. In other words, incomparability relations are "contagious," as they will spread via constant acts. The below axiom **B8** formalizes this idea, and we refer to this form as *contagion of incomparability.*

B8 (Contagion of incomparability)**.** For any distinct acts $f, g \in \mathcal{F}$, if there exists a constant act $x \in \mathcal{F}^c$ such that neither f nor g is comparable with x, then f and g are incomparable.

3.2 Main Theorem

We are now ready to present the main result of this paper.

Theorem 1. *A preference order* $\succsim$ *satisfies* **A1–8** *if and only if it admits an IPOP representation. Moreover,* $C^\sharp$ *and* $C^\flat$ *are unique, and* u *is unique up to positive affine transformations.*

While the formal arguments are relegated to Appendices A and B, we shall provide a proof sketch in the hope that it will be useful to grasp the interplays between axioms and representations. As a

[7] In our axiomatization, **A6** and **A7** can be replaced by the following another weakening of monotonicity:

B6 (Secure-Potential Monotonicity)**.** For any $f, g \in \mathcal{F}$ and $x \in \mathcal{F}^c$, if $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim x$ whenever $g \succsim x$; Also, $x \succsim g$ whenever $x \succsim f$.

It is not hard to see that **B6** imply both **A6** and **A7**, but the converse implications may not be obvious. As such, since any IPOP representation satisfies **B6**, our theorem *indirectly* proves that **A6** and **A7** jointly imply **B6** in the presence of other axioms.

matter of course, a crucial part is the sufficiency of the axioms, which is divided into several steps as below.

First, since **A1–3** assure that $\succsim$ restricted on $\mathcal{F}^c$ is a continuous weak order, we can get a utility function $u : X \to \mathbb{R}$ over outcomes. Then, define a utility function $\overline{U}(f)$ to assign the value of the "minimal" constant act ranked above each general act f. Similarly, $\underline{U}(f)$ is defined to assign the "maximal" constant act ranked below f. Using transitivity, we can show that $\underline{U}(f) \geq \overline{U}(g)$ implies $f \succsim g$. Thus, it remains to prove the reverse direction, i.e., $\succsim$ is *not* more complete than the binary relation jointly represented by $\overline{U}$ and $\underline{U}$. At this point, **A8** plays a key role, as it admits an interpretation as contagious incomparability. Specifically, the axiom adjusts down the degree of $\succsim$'s completeness, so that we have $\underline{U}(f) \geq \overline{U}(g)$ whenever $f \succsim g$ and $f \neq g$. To sum up, the axioms **A1–3** and **A8** derive the following representation: for any $f, g \in \mathcal{F}$, we have

$$f \succsim g \iff \left[\underline{U}(f) \geq \overline{U}(g) \text{ or } f = g\right].$$

At this point, the transitivity of $\succsim$ requires that two utility functions maintain the uniform relation $\overline{U} \geq \underline{U}$.[8]

Second, we derive the integral representations of $\overline{U}$ and $\underline{U}$. After transforming each act into a *utility act* via the mapping $f \mapsto u \circ f$, we naturally define functionals $\overline{T}$ and $\underline{T}$ on utility acts from $\overline{U}$ and $\underline{U}$. Here, **A7** assures that these functionals are well-defined. We then use **A4–6** to provide several properties of $\overline{T}$ and $\underline{T}$, including restricted forms of additivity and monotonicity. Still, the Gilboa and Schmeidler's (1989) representation theorem is not readily applied in the absence of the full monotonicity, so we develop alternative arguments that make use of the shift of contour sets a lá Bewley (2002) to recover belief sets.

To this end, we define upper- and lower-contour sets of each utility act $\xi \equiv u \circ f$ by

$$\mathcal{U}(\xi) = \left\{\zeta : \underline{T}(\zeta) \geq \overline{T}(\xi)\right\} \quad \text{and} \quad \mathcal{L}(\xi) = \left\{\zeta : \underline{T}(\xi) \geq \overline{T}(\zeta)\right\}.$$

These contour sets are then "shifted" to the origin in the two steps as described in Figure 2. Specifically, we first claim that they can be identified with the contour sets of some diagonal elements. In other words, we have $\mathcal{U}(\xi) = \mathcal{U}(\overline{\xi})$ and $\mathcal{U}(\xi) = \mathcal{U}(\underline{\xi})$ for some $\overline{\xi}$ and $\underline{\xi}$ on the 45° line. We then claim that the contour sets of $\overline{\xi}$ and $\underline{\xi}$ are shifted linearly toward the origin, i.e., $\mathcal{U}(\overline{\xi}) = \mathcal{U}(0) + \overline{\xi}$ and $\mathcal{L}(\underline{\xi}) = \mathcal{L}(0) + \underline{\xi}$. These arguments imply that it is enough to characterize $\mathcal{U}(0)$ and $\mathcal{L}(0)$ in order to recover contour sets of general acts. As usual, this final step is done with the help of the separating hyperplane theorem.

To sum up, the main feature of the IPOP representations – the dual-self perspective – is obtained in the first step, where we only impose the axioms **A1–3** and **A8**. As such, **A8** plays a key role in

[8] The way we incorporate reflexivity in our representations may appear to be somewhat ad hoc. At this point, there is the following trade-off between reflexivity and transitivity – while reflexivity requires the uniform relation $\underline{U} \geq \overline{U}$, we must have $\overline{U} \geq \underline{U}$ to maintain transitivity. Hence, having both may imply $\overline{U} = \underline{U}$, but then IPOP reduces to SEU. As reflexivity may be such a basic condition, we have chosen to add an extra rule in the definition of our representation.

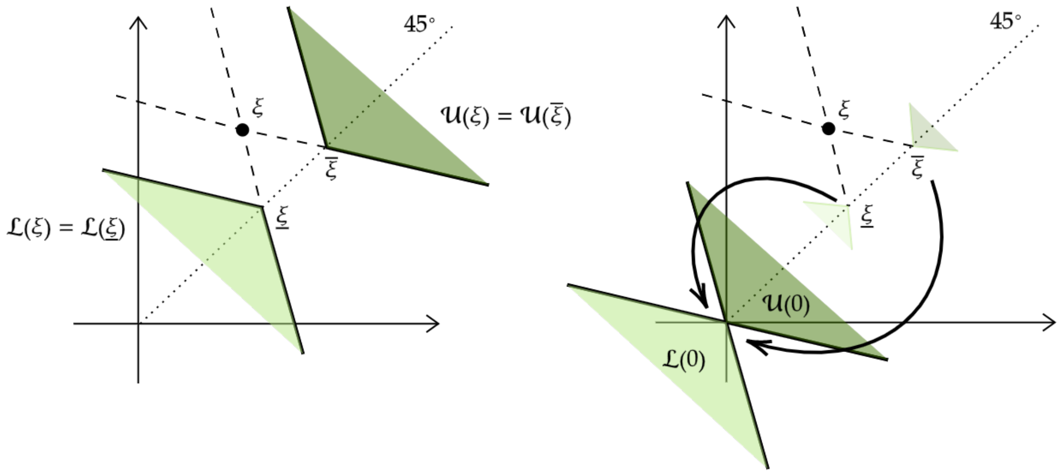

Figure 2: The contour sets of a utility act ξ are shifted to the origin in two steps.

deriving joint representations by two utility functions. Related arguments are utilized by Valenzuela-Stookey (2020) to obtain his simple bound representations.[9] The rest of axioms are used to add geometric properties to utility functions $\overline{U}$ and $\underline{U}$.

We note here that the dual-self perspective is closely related to Fishburn's (1970) interval orders.[10] In a more formal sense, a strict preference order $\succ$ is an interval order if and only if **A1–3** and **A8** are satisfied by its weak part $\succsim$. As such, given $\overline{U}$ and $\underline{U}$ that jointly represent $\succsim$, we define $I(f) = [\underline{U}(f), \overline{U}(f)]$ for each act f, where the transitivity of $\succsim$ assures that $I(f)$ is a well-defined interval when constructed in such as way. In this regard, we obtain some behavioral interpretations of interval orders as a byproduct of the main theorem.

4 Comparative Statics

We study some comparative statics questions with respect to different belief sets. Throughout this section, we shall assume that a preference order $\succsim$ of our interest always admits an IPOP representation

[9] Specifically, his axiom, *uniform comparability*, posits that monotonicity mediated by simple-enough acts is the only inference rule that the DM can use to rank general acts. In contrast, our C-calibration assumes that transitivity mediated by constant acts is the only inference rule that has a similar role.

[10] According to his definition, a strict order $\succ$ on a domain X is an *interval order* if it is irreflexive and satisfies the condition: $x_1 \succ y_1$ and $x_2 \succ y_2$ imply that either $x_1 \succ y_2$ or $x_2 \succ y_1$. Fishburn (1970) shows that any interval relation $\succ$ is represented by a correspondence $I : X \twoheadrightarrow \mathbb{R}$, which outputs a closed interval $I(x) = [a(x), b(x)]$ for each $x \in X$, in the following manner:

- If $I(x) \cap I(y) \neq \emptyset$, then neither $x \succ y$ nor $y \succ x$ (incomparable);
- If $I(x) \cap I(y) = \emptyset$ and $I(x)$ is the right to $I(y)$, then $x \succ y$; and
- If $I(x) \cap I(y) = \emptyset$ and $I(x)$ is the left to $I(y)$, then $y \succ x$.

In other words, an interval order ranks x higher than y if and only if the left-most point of the interval $I(x)$ is greater than the right-most point of $I(y)$.

by some tuple $(u, C^\sharp, C^\flat)$.

4.1 (Non-)equalizable Belief Sets

The definition of IPOP representations still permits belief sets $C^\sharp$ and $C^\flat$ to differ from each other, so naturally one might ask when they coincide. We shall provide an answer to a more general question. Specifically, we detect additional axioms that determine the relative size of one belief sets against one another in terms of set inclusion.

To state our axioms, we introduce the following notion of a perfect hedge between acts, which is closely related to the *complementability* introduced by Siniscalchi (2009).[11]

Definition 2. *An act g is said to be a* perfect hedge against f *if there exists some $\alpha \in (0,1)$ such that $\alpha f(\omega) + (1-\alpha)g(\omega) \sim \alpha f(\omega') + (1-\alpha)g(\omega')$ holds for all $\omega, \omega' \in \Omega$. In such a case, we write as $f \overset{\alpha}{\smile} g$.*

Intuitively, g is a perfect hedge against f if the DM can create a risk-free portfolio by mixing f and g in an appropriate ratio. Any pair of acts that hedges each other cannot be *co-monotonic* according to the definition of Schmeidler (1989), but rather, they move toward "opposite" directions as functions of states. In other words, one act performs better in some states but worse in others, and this trade-off may make the choice between those acts difficult. Hence, if the DM could somehow manage to rank f and g, a choice that involves their mixture might become an easier task, as the mixture is expected to approach a constant act. The next axioms formalize this idea.

A9 (a). For any $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, if $f \succsim g$ and $f \overset{\alpha}{\smile} g$, then $f \succsim \alpha f + (1-\alpha)g$.

A9 (b). For any $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$, if $f \succsim g$ and $f \overset{\alpha}{\smile} g$, then $\alpha f + (1-\alpha)g \succsim g$.

The two axioms differ only in which uncertain prospects, f or g, is comparable with the certain prospect created by them. Specifically, **A9 (a)** states that the DM is more proactive in accepting f against the mixture than in rejecting g against the mixture. **A9 (b)** postulates the exact converse of this. These axioms then characterize the relative size of belief sets, by which we mean the relation of set inclusion between them. As a result, they jointly characterize the situation in which the DM has equal belief sets.

Proposition 1. *Let $\succsim$ be represented by $(u, C^\sharp, C^\flat)$.*

(i) *$C^\sharp \supseteq C^\flat$ if and only if $\succsim$ satisfies* **A9 (a)**.

(ii) *$C^\sharp \subseteq C^\flat$ if and only if $\succsim$ satisfies* **A9 (b)**.

[11] Using the current notation, f and g are *complementable* according to his definition if $f \overset{1/2}{\smile} g$. We remark that the same upshots in Proposition 1 could be obtained even if the axioms **A9 (a–b)** are stated in terms of complementability. As such, the only if directions of this proposition are trivially maintained, and a careful look at the proof reveals that the if directions only use the axioms for $\alpha = 1/2$.

(iii) $C^\sharp = C^\flat$ *if and only if* $\succsim$ *satisfies both* **A9 (a-b)**.

The intuition behind this proposition is explained as follows. Assume that $f \succsim g$ and $f \overset{\alpha}{\smile} g$, and thus, the mixture $\alpha f + (1-\alpha)g$ delivers the same utility in every state. Interpreting the axiom **A8** as stating "contagion of incomparability," the mixture would be comparable with at least one of f or g.[12] As such, **A9 (a)** says that the finer act f is comparable with the mixture whenever g is also comparable with it. This means that the DM is more decisive in evaluating f than how much she is in evaluating g. In a choice between f and g, f is evaluated by the DM's pessimism, while g is evaluated by the DM's optimism. Hence, we conclude that less uncertainty is perceived by the DM's pessimism than optimism, from which we predict $C^\sharp \supseteq C^\flat$. The implication of **A9 (b)** can be understood symmetrically.

4.2 The Amount of Uncertainty

In several existing multiple-prior models, the size of a belief set reflects the amount of uncertainty perceived by the DM, or the degree of completeness of the DM's solid preference order. These ideas have been formalized by Rigotti and Shannon (2005) for Bewley preferences, and by Ghirardato et al. (2004) for MEU. Indeed, an analogous result is also available for IPOP.

Definition 3. *A preference* $\succsim_1$ *is an* extension *of* $\succsim_2$ *if* $f \succsim_1 g$ *whenever* $f \succsim_2 g$*, i.e.,* $\succsim_1 \supseteq \succsim_2$ *when they are viewed as subsets of* $\mathcal{F}^2$*. Furthermore,* $\succsim_1$ *is called a* compatible extension *of* $\succsim_2$ *if* $\succsim_1 \supseteq \succsim_2$*, and* $x \succ_1 y$ *whenever* $x \succ_2 y$ *for any* $x, y \in \mathcal{F}^c$.

Note that since $\succsim_2$ maintains C-completeness, $\succsim_1$ is a compatible extension of $\succsim_2$ if and only if we have both $\succsim_1 \supseteq \succsim_2$ and $\succsim_1|_{\mathcal{F}^c} = \succsim_2|_{\mathcal{F}^c}$.[13]

Proposition 2. *For each* $i \in \{1,2\}$*, let* $\succsim_i$ *admit an IPOP representation by* $(u_i, C_i^\sharp, C_i^\flat)$ *. The followings are equivalent:*

(i) $\succsim_1$ *is a compatible extension of* $\succsim_2$.

(ii) $C_1^\natural \subseteq C_2^\natural$ *for each* $\natural \in \{\sharp, \flat\}$*, and* $u_1 = a u_2 + b$ *for some* $a \in \mathbb{R}_{++}$ *and* $b \in \mathbb{R}$.

5 Connections to Other Decision Models

5.1 Obvious Dominance

IPOP encapsulates SEU and obvious dominance in two polar cases. While the former is pinned down by adding the full completeness axiom, the latter is obtained by adding the full "incompleteness"

[12] This exposition is slightly inaccurate since the axiom requires the mixture to be constant, not only payoff-indifferent. We shall not step into detail here to maintain the intuitiveness of our discussion.

[13] Our definition of compatible extension is slightly different from a popular definition in mathematics, e.g., see Chapter 1 of Aliprantis and Border (2006). Specifically, we require the preservation of strict preferences only on constant acts. This treatment is essential when we characterize the class of all complete, compatible extensions in Section 5.2.2.

axiom presented below. Throughout this subsection, we assume that every singleton is Σ-measurable, i.e., $\{\omega\} \in \Sigma$ for every $\omega \in \Omega$.

A10 (Strong Incomparability). For any $f, g \in \mathcal{F}$, if there exists some $\omega \in \Omega$ such that $g(\omega) \succ f(\omega)$, then $f \not\succsim g$.

This axiom embodies the extreme conservatism of the DM, which urges belief sets to contain all degenerate beliefs.[14]

Proposition 3. *A preference order $\succsim$ satisfies* **A1–8** *and* **A10** *if and only if there exists a non-constant, continuous and affine utility function $u : X \to \mathbb{R}$ such that*

$$f \succsim g \iff \left[\min_{\omega\in\Omega} u(f(\omega)) \geq \max_{\omega\in\Omega} u(g(\omega)) \quad or \quad f = g\right]. \tag{3}$$

Moreover, u is unique up to positive affine transformations.

Zhang and Levin (2017) provide an alternative characterization of obvious dominance within the Anscombe-Aumann's framework. A main difference from this paper is that they maintain the completeness axiom, which makes their characterization somewhat indirect compared with Proposition 3. Specifically, their main result delivers the class of preference orders represented by the following single utility function:

$$U_\alpha(f) = \alpha(f) \max_{\omega\in\Omega} u(f(\omega)) + (1 - \alpha(f)) \min_{\omega\in\Omega} u(f(\omega)),$$

where $\alpha : \mathcal{F} \to [0, 1]$ is an arbitrary function. The authors then argue that obvious dominance is characterized as the "intersection" of preference orders parametrized by several α, while their axioms solely pertain to each U_α. On the other hand, our approach takes incomplete preferences as the model primitive, which allows us to directly axiomatize obvious dominance.

We also note that the class of preference orders obtained by Zhang and Levin (2017) corresponds to the special case of "revealed preferences" that can emerge from IPOP representations. This point will be clarified after we characterize the set of complete extensions of IPOP in the next subsection, cf. Proposition 5.

5.2 Compatible Extensions of IPOP

In several decision problems of interest, the solid preference order of the DM may be incomplete, but a certain decision eventually must be made. As such, one might reasonably ask what decision rules can be obtained from IPOP when an underlying incomplete preference is compatibly extended. In Section 5.2.1, we study the relations among the IPOP and other popular classes of multiple-prior preferences

[14] We remark that **A5** is implied by **A10** under C-independence and C-monotonicity, so the statement of the proposition is actually a bit redundant.

and show that both MEU and Bewley preferences are obtained as compatible extensions of IPOP. In Section 5.2.2, we characterize the whole set of complete preference orders that can be obtained from IPOP.

5.2.1 Multiple-prior Representations

Let us start by clarifying the connections among IPOP and some prominent classes of multiple-prior preferences. The first class is MEU preferences (Gilboa and Schmeidler, 1989), in which a pair of acts are compared in terms of minimal expected values:

$$f \succsim g \Longleftrightarrow \min_{\mu \in C} \int (u \circ f) d\mu \geq \min_{\mu \in C} \int (u \circ g) d\mu,$$

where C is a nonempty, closed and convex set of beliefs. The second class is Bewley preferences (Bewley, 2002), which follows the following unanimity rule that compares a pair of acts in a prior-wise way:

$$f \succsim g \Longleftrightarrow \int (u \circ f) d\mu \geq \int (u \circ g) d\mu, \; \forall \mu \in C.$$

As is well-known, the intersection of these classes solely consists of SEU, which corresponds to the degenerate case in which C is a singleton. On the other hand, Ghirardato et al. (2004) and Gilboa et al. (2010) point out, in different contexts, that each MEU preference can be seen as the complete extension of some Bewley preference. Indeed, there are analogous relations that involve IPOP as well. To formalize this idea, we introduce the following notion of more-conservative relations over different classes of preference orders.

Definition 4. *A nonempty class of preference orders $\mathcal{P}$ is said to be* more conservative *than another nonempty class $\mathcal{P}'$ if*

- *for any $\succsim \in \mathcal{P}$ there exists some $\succsim' \in \mathcal{P}'$ such that $\succsim'$ is a compatible extension of $\succsim$; and*
- *for any $\succsim' \in \mathcal{P}'$ there exists some $\succsim \in \mathcal{P}$ such that $\succsim'$ is a compatible extension of $\succsim$.*

To rephrase the previous observation, the class of Bewley preferences ($\mathcal{P}_{\text{Bewley}}$) is more conservative than the class of MEU preferences ($\mathcal{P}_{\text{MEU}}$). The next proposition finds that the class of IPOP preferences ($\mathcal{P}_{\text{IPOP}}$) is actually more conservative than both $\mathcal{P}_{\text{MEU}}$ and $\mathcal{P}_{\text{Bewley}}$. On the other hand, the intersection of $\mathcal{P}_{\text{IPOP}}$ and $\mathcal{P}_{\text{MEU}}$, as well as the intersection of $\mathcal{P}_{\text{IPOP}}$ and $\mathcal{P}_{\text{Bewley}}$, consists solely of the class of SEU preferences ($\mathcal{P}_{\text{SEU}}$). This is because an IPOP representation satisfies monotonicity only when its belief sets are the same singleton.

Proposition 4. (i) $\mathcal{P}_{\text{MEU}} \cap \mathcal{P}_{\text{Bewley}} = \mathcal{P}_{\text{SEU}}$*, while $\mathcal{P}_{\text{Bewley}}$ is more conservative than $\mathcal{P}_{\text{MEU}}$.*

(ii) $\mathcal{P}_{\text{MEU}} \cap \mathcal{P}_{\text{IPOP}} = \mathcal{P}_{\text{SEU}}$*, while $\mathcal{P}_{\text{IPOP}}$ is more conservative than $\mathcal{P}_{\text{MEU}}$.*

(iii) $\mathcal{P}_{\text{Bewley}} \cap \mathcal{P}_{\text{IPOP}} = \mathcal{P}_{\text{SEU}}$*, while $\mathcal{P}_{\text{IPOP}}$ is more conservative than $\mathcal{P}_{\text{Bewley}}$.*

5.2.2 Completions of IPOP

We say that a preference order $\succsim^*$ is a *completion* of an IPOP order $\succsim$ if it is a compatible extension of $\succsim$ and satisfies both transitivity and completeness. By Proposition 4, any IPOP order has some MEU preference as its completion, but MEU does not cover the whole class of IPOP's completions. In the next proposition, we provide the full characterization of IPOP's completions that maintain one weak continuity condition.[15]

Proposition 5. *Let $\succsim$ be a preference order that admits an IPOP representation by a tuple $(u, C^\sharp, C^\flat)$. A preference order $\succsim^*$ is a C-continuous completion of $\succsim$ if and only if $\succsim^*$ is represented by a utility function $I : \mathcal{F} \to \mathbb{R}$ taking the following form:*

$$I(f) = \alpha(f) \min_{\mu \in C^\flat} \int (u \circ f) d\mu + (1 - \alpha(f)) \max_{\mu \in C^\sharp} \int (u \circ f) d\mu, \tag{4}$$

where $\alpha : \mathcal{F} \to [0, 1]$ is an arbitrary function.

We refer to the class of preference orders represented in the form (4) as *generalized α-maximin preferences.* This class is general in that a fraction $\alpha(\cdot)$ can vary across different acts, which permits the possibility for $\succsim^*$ to violate independence or monotonicity. Its important subclass includes α-maximin preferences, and their generalization such as invariant biseparable preferences (Ghirardato et al., 2004; Chandrasekher et al., 2020). Note that no structure of $\alpha(\cdot)$ is needed to ensure the C-continuity of $\succsim^*$ as the expected utility of a constant act is independent of beliefs. In other words, $\alpha(\cdot)$ is not identified on $\mathcal{F}^c$.

Proposition 5 builds a bridge between IPOP, which is held in the DM's mind, and its completion, which is observable as a revealed choice. The relation between them would be well understood in line with Gilboa et al. (2010), who introduce the notions of "objective rationality" and "subjective rationality" in choice formation processes.[16] Their expositions can be rearranged to be accommodated in the current context. An IPOP order $\succsim$ reflects choices made by the DM that are rational in an "obviously objective" sense: The DM has proof that she is right in making them even when other people may not share a common sense of the states of the world. On the other hand, when she cannot perform contingent reasoning, a proof that sounds compelling to her should be simple enough that it does not involve state-by-state considerations. A completion $\succsim^*$, therefore, reflects choices that are rational in "obviously subjective" sense: There is no proof that convinces the DM that she is wrong in making them without using contingent reasoning.

With the above proposition in hand, a specific completion rule that extends an IPOP order to the corresponding maximin or maximax preferences can be derived without deficulty. To state these

[15] Allowing for the violation of C-continuity, there emerge lexicographic-type completions that are not represented in generalized α-maximin forms as in this proposition. See Appendix D.2 for a concrete example.

[16] According to their definitions, a choice is objectively rational if the DM has proof that she is right in making it, whereas a choice is subjectively rational if others do not have proof that she is wrong in making it. Gilboa et al. (2010) use these notions to provide a joint characterization of MEU and Bewley preferences.

results, let $\succsim$ be a preference order that admits an IPOP representation by a tuple $(u, C^\sharp, C^\flat)$, and assume that $C \equiv C^\sharp = C^\flat$. Further, let $\succsim^*$ be a C-continuous completion of $\succsim$. Following the literature, we say that $\succsim$ and $\succsim^*$ jointly satisfy *caution* if for any $f \in \mathcal{F}$ and $x \in X$, $f \not\succsim x$ implies that $x \succsim^* f$. We also say that $\succsim$ and $\succsim^*$ jointly satisfy *abandon* if for any $f \in \mathcal{F}$ and $x \in X$, $x \not\succsim f$ implies that $f \succsim^* x$.

Corollary 1. *A pair $(\succsim, \succsim^*)$ jointly satisfies caution (resp. abandon) if and only if $\succsim^*$ admits the maximin (resp. maximax) representation by (u, C).*

6 Application to Second-price Auctions

Economists have encountered a "gap" between theory and empirics in second-price auctions. Contrary to theoretical predictions, much empirical evidence documents bidders who do not report their true valuations. To provide a clue to understanding this gap, Li (2017) proposes the concept of obvious dominance. While his decision rule can explain the departure from truth-telling by assuming a bidder is "extremely" conservative, the rule is too strong to generate a reasonable range of predictions, i.e., any bidding strategy in a second-price auction is not dominated in terms of obvious dominance.[17]

In this section, we apply our decision-theoretic model to revisit second-price auctions. As such, a bidder is too conservative to have a unique belief about opponents' bids but not as conservative to apply Li's decision rule with respect to all degenerate beliefs. We show that for generic belief sets, there appear a continuum of undominated bids, meaning that the departure from truth-telling can be explained by parsimoniously assuming that a bidder is "slightly" conservative. On the other hand, we can eliminate deviating strategies far enough from a true valuation when belief sets are reasonably smaller than all degenerate beliefs. Hence, our approach can provide sharper predictions about what deviations are more likely to occur than others. Lastly, we try to shed some light on an empirical question of why overbidding is more commonly observed than underbidding, e.g., Kagel et al. (1987) and Kagel and Levin (1993).

6.1 The Model

A single indivisible good is auctioned off among potential buyers. We fix an arbitrary buyer (DM) among them and study her undominated bidding strategies in a second-price auction. The personal outcome space is given by $(x, t) \in [0, 1] \times [0, \bar{b}] \equiv X$, where x is a probability of obtaining the good, t is payment to the auctioneer, and $\bar{b}$ is the maximal payment exogenously set by the auction platform. Assume that the DM evaluates each outcome (x, t) by the ex-post utility function $u(x, t)$, which is strictly increasing in x, and strictly decreasing and continuous in t. Further, we normalize $u(0, 0) = 0$,

[17] Recall that obvious dominance is characterized within IPOP by adding an extra axiom **A10**, which postulates the DM's extreme conservatism. Meanwhile, it should be emphasized that Li's (2017) main interest is not in explaining the departure from truth-telling but rather in clarifying a practical difference in extensive mechanisms that share the same normal form. It is fair to say that Li's attempt has succeeded greatly in this regard.

therefore, there exists a unique value $v \in (0, \bar{b})$ that satisfies $u(1, v) = u(0, 0)$. We refer to v as the DM's valuation. Note that $u(x, t) = vx - t$ when u is quasi-linear. Let us fix a utility function u throughout this section.

We consider the following reduced form to model uncertainty faced by the DM. Note that in a second-price auction, only the highest opponent's bid can affect the DM's payoff. Thus, a single-dimensional space $\Omega \equiv [0, \bar{b}]$ summarizes all payoff-relevant uncertainty, and a second-price auction induces the DM's personal outcomes by

$$(x(b, \omega), t(b, \omega)) = \begin{cases} (1, \omega) & \text{if } b \geq \omega, \\ (0, 0) & \text{if } b < \omega. \end{cases}$$

Here, we assume that ties are broken in favor of the DM, but this assumption is without loss of generality if the DM's beliefs are non-atomic.

When the DM conceives that ω is distributed according to $\mu \in \Delta(\Omega)$, her subjective expected utility is simply given as follows:[18]

$$U(b, \mu) \equiv \int_0^{\bar{b}} u(x(b, \omega), t(b, \omega)) d\mu(\omega) = \int_0^b u(1, \omega) d\mu(\omega).$$

A prominent feature of a second-price auction is that $b^* = v$ maximizes $U(\cdot, \mu)$ regardless of μ, and b^* is a unique maximizer if there is a neighborhood of v on which μ is fully supported. In other words, the truth-telling is generally a unique undominated strategy when DM's uncertainty is captured by a single subjective belief.

The unique optimality of truth-telling is no longer assured when there are multiple beliefs, and the DM is bounded rational in the sense of our decision model. Specifically, assume that the DM conceives non-disjoint, closed, and convex belief sets $C^\sharp, C^\flat \subseteq \Delta(\Omega)$, and she evaluates bidding strategies based on the corresponding IPOP representation. The set of undominated bidding strategies can then be naturally defined in terms of the IPOP representation:

$$B^*(C^\sharp, C^\flat) = \left\{ b \in [0, \bar{b}] : \nexists b' \in [0, \bar{b}] \setminus \{b\} \ \text{ s.t. } \min_{\mu \in C^\flat} \int_0^{b'} u(1, \omega) d\mu(\omega) \geq \max_{\mu \in C^\flat} \int_0^b u(1, \omega) d\mu(\omega) \right\}.$$

6.2 Undominated Bids

For the sake of tractability, we impose the following regularity conditions.

Condition 1. *Every $\mu \in C^\sharp \cup C^\flat$ is a non-atomic probability measure on (Ω, Σ).*[19]

Condition 2. *There exists some $\mu \in C^\sharp \cap C^\flat$ that is fully supported on a neighborhood of v.*

[18] Unless otherwise specified, integrations are performed on closed intervals.

[19] A probability measure μ is *non-atomic* if for any event $A \in \Sigma$ with $\mu(A) > 0$, there exists an even $B \subseteq A$ such that $\mu(A) > \mu(B) > 0$.

As is well-known, under Condition 1, each belief μ has a continuous density function, which we may write as ϕ_μ. Condition 2 ensures that there exists at least one belief μ according to which truth-telling becomes a unique optimal strategy.

We say that *truth-telling is transparent* for the DM if the optimistic and pessimistic expected utilities from the truth telling strategy $b^* = v$ are the same, i.e.,

$$\max_{\mu \in C^\sharp} \int_0^v u(1,\omega) d\mu(\omega) = \min_{\mu \in C^\flat} \int_0^v u(1,\omega) d\mu(\omega). \tag{5}$$

The next proposition reveals the structure of an undominated set. An undominated set can be a singleton if and only if truth-telling is transparent. In particular, as formally discussed in Appendix D.3, the equality (5) is a knife-edge condition, meaning that truth-telling is not transparent for "almost every" pair of belief sets. Hence, for generic belief sets, there emerges a continuum of deviating bids not dominated by any other bids.

Proposition 6. *Suppose that Condition 1 is satisfied.*

(i) *If truth-telling is transparent, then $B^*(C^\sharp, C^\flat) \subseteq \{v\}$. In particular, if Condition 2 is satisfied, then $B^*(C^\sharp, C^\flat) = \{v\}$.*

(ii) *If truth-telling is not transparent, then $B^*(C^\sharp, C^\flat)$ is an open interval that contains v.*[20]

The intuition behind Proposition 6 can be understood from Figure 3. Here, the DM possesses three beliefs, say μ_1, μ_2, and μ_3, according to which the DM's expected utility is drawn as a function of her bid. These functions vary across beliefs, but we know that each expected utility function is increasing on $[0, v)$ while decreasing on $(v, \bar{b}]$.[21] Namely, all these functions are single-peaked around v. Importantly, the same property is inherited to the minimal expected utility so that the blue line in the figure maintains single-peakedness as well. Hence, to check whether a given bid b belongs to the undominated set, all we need is to check whether b's maximal expected utility exceeds v's minimal expected utility, marked with a blue star above. Further, the maximal expected utility inherits single-peakedness as well, so the region in which the red line exceeds the peak of the blue line must be an interval. In particular, the undominated interval has a nonempty interior provided that the peaks of red and blue lines are different, i.e., the transparency condition is not met.

Thus, in second-price auctions, the departure from truth-telling can be explained by the parsimonious assumption that participants are "slightly" bounded rational, rather than they are "extremely" bounded rational as postulated by Li (2017). As such, obvious dominance may assume too much – only the trivial prediction set would be available when the DM's belief sets are too large.

Proposition 7. *If both belief sets are consist of the all probability measures on (Ω, Σ), then $B^*(C^\sharp, C^\flat) = [0, \bar{b}]$.*

[20] The topological notion is relative to $[0, \bar{b}]$.

[21] This is due to the fact that v is a dominant strategy in the usual sense, and thus, optimal when the DM's uncertainty is governed by any single belief.

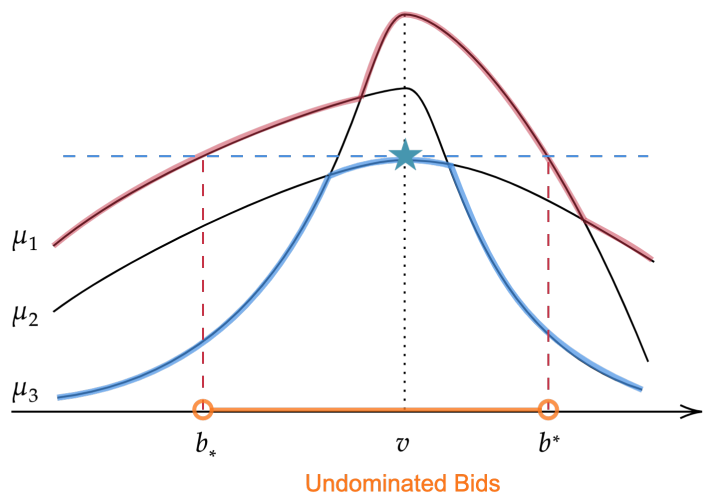

Figure 3: The intuition behind Proposition 6

Proof. Consider a belief μ_0 that assigns a mass to a point $0 \in \Omega$, according to which every bid b trivially achieves the maximal expected utility of $u(1, 0)$. (Note that $b = 0$ can also achieve $u(1, 0)$ because ties are broken in favor of the DM.) On the other hand, v's minimal expected utility is strictly less than $u(1, 0)$ since $C^\flat$ contains every probability measure. Thus, any b is undominated, thereby $B^*(C^\sharp, C^\flat)$ coincides with $[0, \bar{b}]$. $\square$

Our comparative statics result (Proposition 2) is readily translated to undominated bids in second-price auctions.

Corollary 2. *Suppose that Condition 1 is satisfied by* $(C_i^\sharp, C_i^\flat)$ *for* $i \in \{1, 2\}$*. If* $C_1^\natural \subseteq C_2^\natural$ *for each* $\natural \in \{\sharp, \flat\}$*, then* $B^*(C_1^\sharp, C_1^\flat) \subseteq B^*(C_2^\sharp, C_2^\flat)$*.*

Proof. By Proposition 2, we know that the incomplete preference $\succsim_1$ defined by $(C_1^\sharp, C_1^\flat)$ is a subset of $\succsim_2$, which is defined by $(C_2^\sharp, C_2^\flat)$. Thus, if $\succsim_1$ does not rank v higher than b, then so does $\succsim_2$. The proof is done with this and (i) of Proposition 6. $\square$

6.3 Overbidding vs Underbidding

There is robust empirical evidence that subjects are more likely to exhibit a consistent pattern of overbidding, rather than underbidding; however, there is little theory that explains the reason behind it.[22] Among some available candidate explanations, Kagel et al. (1987) infer that overbidding is likely based on the illusion that it improves the chance of winning with no real cost to the winner as only the second-highest bid is paid, whereas underbidding substantially decreases the chance of winning

[22] For example, in Kagel et al. (1987), the actual bids submitted by subjects are, on average, 11% higher than theoretical predictions. In Kagel and Levin (1993), while only 8% of bids fall below the true valuations, about 62% of bids exceed the true valuations.

with no real benefit from reducing the payment. In other words, they conjecture that overbidding can be partly attributed to asymmetry in the rules of second-price auctions.

However, in the current decision model, we see that the structural asymmetry of second-price auctions *per se* is not enough to explain the empirical tendency toward overbidding. Specifically, the next proposition indicates that undominated bids would be symmetrically located around the true valuation, provided that the DM's tastes and beliefs are symmetric.[23] Hence, to generate an asymmetric range of undominated bids, we must seek explanations for the DM's primitives, rather than the mechanism itself.

Proposition 8. *Suppose that Condition 1 is satisfied. Furthermore, suppose that $u(x,t) = vx - t$ is quasi-linear, and every $\mu \in C^\sharp$ satisfies $\phi_\mu(v+t) = \phi_\mu(v-t)$ whenever $v \pm t \in [0, \bar{b}]$. Then, for any such t, we have $v + t \in B^*(C^\sharp, C^\flat)$ if and only if $v - t \in B^*(C^\sharp, C^\flat)$.*

Proof. Fix any t such that $v \pm t \in [0, \bar{b}]$. By Proposition 6, it is enough to show that $\overline{U}(v+t) = \overline{U}(v-t)$, where $\overline{U} = \max_{\mu \in C^\sharp} U(\cdot, \mu)$. Indeed, even a sufficient condition for this is true: We show that $\overline{U}(v+t,\mu) = \overline{U}(v-t,\mu)$ for every $\mu \in C^\sharp$. By the quasi-linearity, we have

$$\overline{U}(v+t,\mu) - \overline{U}(v-t,\mu) = \int_{v-t}^{v+t} (v-\omega)\phi_\mu(\omega)d\omega = -\int_{-t}^{t} \omega\phi_\mu(\omega - v)d\omega,$$

where we perform the change of variables $\omega \mapsto \omega - v$. Since a function $\omega \mapsto \omega$ is odd, and $\omega \mapsto \phi_\mu(\omega - v)$ is even due to the ϕ_μ's symmetry around v, the above integral is zero. □

There are at least two ways that we can depart from Proposition 8 to generate an asymmetric range of undominated bids. First, and most obviously, we could assume that $C^\sharp$ contains asymmetric beliefs. Second, we can depart from the assumption of quasi-linearity that has generated the symmetric ex-post payoffs around the truth-telling strategy. This would be plausible in auctions for expensive items, whose income effects are not negligible.

7 Conclusion

In the presence of uncertainty, the DM must think contingently to properly understand the consequences of her choice. In some contexts, this type of reasoning is too difficult for an average agent, as documented by a large body of empirical literature. Motivated by these empirical findings, this paper has studied the implications of weakening the standard postulates in decision theory. Specifically, we have weakened the Anscombe-Aumann's core axioms to hold only when constant acts are involved. As a sequel, we have obtained the class of preference orders that are more conservative than those most popular in the literature of decision making under uncertainty.

[23] Note that the result below does not address the "frequency" of misreporting in a specific direction, but rather pertains only to the "magnitude".

What are the main contributions of this paper? In our view, one major contribution is that the paper introduces the novel representation that continuously fill the gap between two decision criteria – expected utility maximization and obvious dominance – that respectively provide foundations for Bayesian incentive compatibility and obvious strategy-proofness in mechanism design. Presumably, the former is one of the least demanding incentive constraints, whereas the latter is the most demanding of those proposed so far. The use of a stronger concept can improve the robustness of mechanisms against misplays by participants. At the same time, however, nonexistence issues may arise, i.e., there may not exist a mechanism that fulfills strong incentive constraints. Indeed, obviously strategy-proof mechanisms can exist in only limited economic environments. In this regard, our axiomatic analysis of bounded rationality may be useful to indicate possible directions to which incentive constraints should be weakened in the hope of making a better compromise.

An interesting extension is the following generalized dual-self representations – in the general form of our representation result (Lemma 1), the relevant axioms are stated without the affine structure of the outcome space. In this regard, we conjecture that we could start our analysis with an abstract topological space $\mathcal{F}$ of acts, and its subset $\mathcal{F}^c \subseteq \mathcal{F}$ interpreted as the collection of acts somewhat "clearly" understood by the DM. Generalizing the model in this way leads to exploring various new applications. For example, such modeling may be useful to study the *framing effect* (Tversky and Kahneman, 1981) by allowing us to make an explicit distinction between acts that are materially the same but with different connotations, expositions, or descriptions.[24] Given the recent focus on the simplicity of mechanism design, these issues await further inspection.

Appendix A General Representation Results

Before proving Theorem 1, we provide a benchmark result which are stated in terms of general representation forms. Specifically, we show that a part of our axioms suffice to represent $\succsim$ jointly by abstract utility functions $\overline{U}$ and $\underline{U}$.

Given any function $U : \mathcal{F} \to \mathbb{R}$, we denote its image by $\mathrm{Im}(U) = \{U(f) \in \mathbb{R} : f \in \mathcal{F}\}$.

Lemma 1. *A preference relation $\succsim$ satisfies* **A1–3** *and* **A8** *if and only if there exist non-constant functions $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ with $\overline{U} \geq \underline{U}$ such that*

(i) $\overline{U}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$ *holds, and the restriction is continuous on $\mathcal{F}^c$;*

(ii) $\mathrm{Im}(\overline{U}|_{\mathcal{F}^c}) = \mathrm{Im}(\overline{U}) = \mathrm{Im}(\underline{U}) = \mathrm{Im}(\underline{U}|_{\mathcal{F}^c})$*; and*

(iii) $f \succsim g$ *if and only if* $\underline{U}(f) \geq \overline{U}(g)$ *or* $f = g$.

[24] For example, in the current model, the DM perceives two acts in the same way whenever they are identical as functions from states to outcomes. Thus, $f(\omega) = \frac{1}{\cos^2(\omega)} - \tan^2(\omega)$ and 1 are treated as the same object, but the former is perhaps less easily recognized as a constant act.

Moreover, a pair of functions $\overline{V}, \underline{V} : \mathcal{F} \to \mathbb{R}$ satisfies (i)-(iii) for the same preference relation $\succsim$ if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\overline{V} = \phi \circ \overline{U}$ and $\underline{V} = \phi \circ \underline{U}$.

Proof. The crucial part is the sufficiency of axioms. Suppose that $\succsim$ satisfies **A1–3** and **A8**. Since the restriction $\succsim|_{\mathcal{F}^c}$ is a continuous weak order on $\mathcal{F}^c \simeq X$ by **A1–3**, there exists a continuous function $u : X \to \mathbb{R}$ that represents $\succsim|_{\mathcal{F}^c}$. Define $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ by

$$\overline{U}(f) = \min\big\{u(x) : x \in \mathcal{U}^c(f)\big\}, \tag{6}$$

$$\underline{U}(f) = \max\big\{u(x) : x \in \mathcal{L}^c(f)\big\}, \tag{7}$$

where

$$\mathcal{U}(f) = \{g \in \mathcal{F} : g \succsim f\}, \quad \mathcal{U}^c(f) = \mathcal{U}(f) \cap \mathcal{F}^c, \tag{8}$$

$$\mathcal{L}(f) = \{g \in \mathcal{F} : f \succsim g\}, \quad \mathcal{L}^c(f) = \mathcal{L}(f) \cap \mathcal{F}^c. \tag{9}$$

Since both $\mathcal{U}^c(f)$ and $\mathcal{L}^c(f)$ are nonempty compact sets, these utility functions are well-defined. Clearly, $u = \overline{U}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$, and the restriction u is continuous on the space $\mathcal{F}^c \simeq X$.

Let us show that $f \succsim g$ if and only if $\underline{U}(f) \geq \overline{U}(g)$ or $f = g$. Starting with the if direction, reflexivity implies that $f \succsim g$ whenever $f = g$. Suppose that $\underline{U}(f) \geq \overline{U}(g)$. Then, there are outcomes $x, y \in X$ such that $u(x) = \underline{U}(f) \geq \overline{U}(g) = u(y)$. Since u represents $\succsim$, we have $x \succsim y$, while $f \succsim x$ and $y \succsim g$ hold by the constructions of x and y. Using transitivity twice, we get $f \succsim g$ as required. Conversely, suppose that $\underline{U}(f) < \overline{U}(g)$ and $f \neq g$. Again, let $x, y \in X$ be outcomes that attain the values of $\underline{U}(f)$ and $\overline{U}(g)$, respectively. Suppose not, $f \succsim g$ holds. By **A8**, there exists $z \in \mathcal{F}^c$ for which $f \succsim z$ and $z \succsim g$, meaning that $z \in \mathcal{L}^c(f)$ and $z \in \mathcal{U}^c(g)$. Since x maximizes u in $\mathcal{L}^c(f)$, it follows that $u(x) \geq u(z)$. Similarly, by the minimality of y in $\mathcal{U}^c(g)$, we must have $u(z) \geq u(y)$. Therefore, it follows that $u(x) \geq u(y)$, but this is a contradiction to that $u(x) = \underline{U}(f) < \overline{U}(g) = u(y)$. Hence, $f \not\succsim g$ must hold.

Finally, we shall prove the uniqueness part. The if direction of the statement is easy to verify. Now, suppose that $(\overline{U}, \underline{U})$ and $(\overline{V}, \underline{V})$ satisfy (i)–(iii) of Lemma 1 for the same preference $\succsim$. In particular, the restrictions $u \equiv \overline{U}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$ and $v \equiv \overline{V}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$ represent the same weak order $\succsim|_{\mathcal{F}^c}$ on $\mathcal{F}^c \simeq X$, and hence, each must be a monotonic transformation of one another, i.e., there exists a strictly increasing function $\phi : \mathrm{Im}(u) \to \mathbb{R}$ for which $v = \phi \circ u$. Note that ϕ must be continuous to maintain continuity of u and v. Fix any non-constant act $f \in \mathcal{F}$. By $\mathrm{Im}(u) = \mathrm{Im}(\overline{U})$, there exists $\overline{x}_f$ such that $u(\overline{x}_f) = \overline{U}(f)$. Since $(\overline{U}, \underline{U})$ represents $\succsim$,

$$\mathcal{U}(f) \setminus \{f\} = \{g \in \mathcal{F} \setminus \{f\} : \underline{U}(g) \geq \overline{U}(f)\} = \{g \in \mathcal{F} \setminus \{f\} : \underline{U}(g) \geq u(\overline{x}_f)\} = \mathcal{U}(\overline{x}_f) \setminus \{f\},$$

where $\mathcal{U}(\cdot)$ is a contour set defined by $\succsim$. On the other hand, repeating the same argument for $(\overline{V}, \underline{V})$, we can there find $\overline{y}_f$ such that $v(\overline{y}_f) = \overline{V}(f)$, which in turn yields

$$\mathcal{U}(f) \setminus \{f\} = \{g \in \mathcal{F} \setminus \{f\} : \underline{V}(g) \geq \overline{V}(f)\} = \{g \in \mathcal{F} \setminus \{f\} : \underline{U}(g) \geq u(\overline{y}_f)\} = \mathcal{U}(\overline{y}_f) \setminus \{f\}.$$

Hence, $\mathcal{U}(\overline{x}_f) \setminus \{f\} = \mathcal{U}(\overline{y}_f) \setminus \{f\}$ holds. In particular, this implies $\mathcal{U}^c(\overline{x}_f) = \mathcal{U}^c(\overline{y}_f)$, which can be true only when $u(\overline{x}_f) = u(\overline{y}_f)$. Therefore, we have

$$\phi \circ \overline{U}(f) = \phi \circ u(\overline{x}_f) = \phi \circ u(\overline{y}_f) = v(\overline{y}_f) = \overline{V}(f),$$

as desired. The symmetric argument verifies that $\phi \circ \underline{U} = \underline{V}$. □

We say $U : \mathcal{F} \to \mathbb{R}$ is *C-affine* if $U(\alpha f + (1-\alpha)x) = \alpha U(f) + (1-\alpha)U(x)$ for all $f \in \mathcal{F}$, $x \in \mathcal{F}^c$, and $\alpha \in (0,1)$. Also, a real function $u : X \to \mathbb{R}$ is *affine* if $u(\alpha x + (1-\alpha)y) = \alpha u(x) + (1-\alpha)u(y)$ for all $x, y \in X$ and $\alpha \in (0,1)$. Note that if U is C-affine, then the restriction $U|_{\mathcal{F}^c}$ is affine on the domain $\mathcal{F}^c$ that is isomorphic to X. The next lemma states that the axiom **A4** makes $\overline{U}$ and $\underline{U}$ C-affine, which in turn makes $u = \overline{U}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$ affine.

Lemma 2. *Suppose that $\succsim$ satisfies* **A1–3** *and* **A8**. *Then, $\succsim$ satisfies* **A4** *if and only if there exist non-constant and C-affine functions $\overline{U}$ and $\underline{U}$ that represent $\succsim$ in the way of Lemma 1.*

Proof. It is easy to see that $\succsim$ satisfies **A4** whenever $\overline{U}$ and $\underline{U}$ are C-affine. Conversely, suppose that $\succsim$ satisfies **A4**. Since $\succsim|_{\mathcal{F}^c}$ maintains all the vNM axioms, there exists a continuous affine function $u : X \to \mathbb{R}$ that represents $\succsim|_{\mathcal{F}^c}$. Again, let $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ be defined by (6) and (7), so that $\overline{U}$ and $\underline{U}$ jointly represent $\succsim$ in the way that Lemma 1 prescribes.

Since the argument is symmetric, we only prove that $\overline{U}$ is C-affine. Fix any $f \in \mathcal{F}$, $y \in \mathcal{F}^c$, and $\alpha \in (0,1)$. Let $x \in \mathcal{U}^c(f)$ be a constant act such that $\overline{U}(f) = u(x)$. By the construction, $x \succsim f$ holds, so that **A4** yields $\alpha x + (1-\alpha)y \succsim \alpha f + (1-\alpha)y$. Since $\overline{U}$ and $\underline{U}$ jointly represent $\succsim$, we have

$$\overline{U}(\alpha f + (1-\alpha)y) \leq \underline{U}(\alpha x + (1-\alpha)y) = u(\alpha x + (1-\alpha)y) = \alpha u(x) + (1-\alpha)u(y),$$

where the equalities follow from $\underline{U}|_{\mathcal{F}^c} = u$ and the affinity of u.

To show that the above inequality is tight, we first claim that x is on the boundary of $\mathcal{U}^c(f) \subseteq X$. If not, we could have an open ball of x which is contained in $\mathcal{U}^c(f)$, but then, there must exist some $\tilde{x} \in \mathcal{U}^c(f)$ with $u(\tilde{x}) < u(x)$ because u is affine and non-constant. This leads to a contradiction to the minimality of x in $\mathcal{U}^c(f)$. Hence, by the continuity of u, for an arbitrarily small $\epsilon > 0$, we can pick $x_\epsilon \in X \setminus \mathcal{U}^c(f)$ such that $|u(x) - u(x_\epsilon)| < \epsilon$. In particular, since $\overline{U}$ and $\underline{U}$ represent $\succsim$, we have $u(x) = \underline{U}(f) > u(x_\epsilon)$, from which $0 < u(x) < \epsilon + u(x_\epsilon)$. Moreover, **A4** and $x_\epsilon \not\succsim f$ together imply that $\alpha x_\epsilon + (1-\alpha)y \not\succsim \alpha f + (1-\alpha)y$. Again by the fact that $\overline{U}$ and $\underline{U}$ represent $\succsim$, we have

$\overline{U}(\alpha f + (1-\alpha)y) \geq u(\alpha x_\epsilon + (1-\alpha)y)$. To sum up,

$$\begin{aligned}\alpha u(x) + (1-\alpha)u(y) &< \alpha u(x_\epsilon) + (1-\alpha)u(y) + \alpha\epsilon \\ &= u(\alpha x_\epsilon + (1-\alpha)y) + \alpha\epsilon \leq \overline{U}(\alpha f + (1-\alpha)y) + \alpha\epsilon.\end{aligned}$$

Letting $\epsilon \to 0$ yields the desired inequality. □

Appendix B Proof of Theorem 1

We omit the trivial proof of necessity. We proves the sufficiency of the axioms in Step 1, and the uniqueness part in Step 2.

Step 1: Sufficiency.

Suppose that $\succsim$ satisfies **A1–8**. By Lemmas 1 and 2, the axioms **A1–4** and **A8** imply that there exist C-affine utility functions $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ that represent $\succsim$ in the way that these lemmas describe. Specifically, there exists a non-constant, continuous and affine function $u : X \to \mathbb{R}$ that calibrate $\overline{U}$ and $\underline{U}$ in the way of (6) and (7), respectively. Henceforth, we arbitrarily fix such a function u, and assume without loss of generality that $[-1, 1] \subseteq \mathrm{Im}(u)$ by means of non-degeneracy.

Define $\Xi = \{u \circ f : f \in \mathcal{F}\} \subseteq (\mathrm{Im}(u))^\Omega$. A generic element $\xi \in \Xi$ is a Σ-measurable real function on Ω and called a *utility act*, interpreted as a profile of utility values carried by an act. Since u is affine, Ξ is a convex subset of $\mathbb{R}^\Omega$. Note that $\{\xi \in [-1, 1]^\Omega : \xi \text{ is } \Sigma\text{-measurebale}\} \subseteq \Xi$ by the normalization of u. Then, define functionals $\overline{T}, \underline{T} : \Xi \to \mathbb{R}$ by

$$\overline{T}(\xi) = \overline{U}(f) \quad \text{and} \quad \underline{T}(\xi) = \underline{U}(f), \quad \text{where } \xi = u \circ f.$$

Note that **A7** assures that these functionals are well-defined, i.e., $\overline{T}(\xi)$ and $\underline{T}(\xi)$ are uniquely determined regardless of the choice of acts f and g that deliver the same utility act.

Denote by $\mathcal{B}(\Omega, \Sigma)$, or simply by $\mathcal{B}$, the set of all bounded Σ-measurable real functions, and let $\mathcal{B}^c = \{\xi \in \mathcal{B} : \xi(\omega) = \xi(\omega') \text{ for all } \omega, \omega' \in \Omega\}$ be the set of diagonal elements in $\mathcal{B}$. Also, let $\mathcal{B}_+ = \mathcal{B} \cap \mathbb{R}^\Omega_+$ and $\mathcal{B}_- = \mathcal{B} \cap \mathbb{R}^\Omega_-$. As usual, $(\mathcal{B}, \|\cdot\|_\infty)$ becomes a Banach space endowed with the sup norm $\|\xi\|_\infty = \sup_{\omega\in\Omega} |\xi(\omega)|$. Note that $\Xi \subseteq \mathcal{B}$ since u is continuous on the compact domain X. Let $\mathbf{1}_\Omega : \Omega \to \mathbb{R}$ denote a constant function that takes 1 for every $\omega \in \Omega$, so that $\lambda\mathbf{1}_\Omega$ denotes the one that takes a real $\lambda \in \mathbb{R}$ for every $\omega \in \Omega$.

Now, we shall show that $\overline{T}$ and $\underline{T}$ are positively homogeneous, and hence, these functionals can be uniquely extended to cover the whole space $\mathcal{B}$.

Claim 1. $\overline{T}(\lambda\mathbf{1}_\Omega) = \underline{T}(\lambda\mathbf{1}_\Omega) = \lambda$ *for any* $\lambda \in \mathrm{Im}(u)$.

Proof. Given any $\lambda \in \mathrm{Im}(u)$, let $x \in X$ be an outcome such that $u(x) = \lambda$. By construction, $\lambda\mathbf{1}_\Omega = u \circ x\mathbf{1}_\Omega$, and hence, $\overline{T}(\lambda\mathbf{1}_\Omega) = \overline{U}(x) = u(x) = \lambda$. Similarly, we have $\underline{T}(\lambda\mathbf{1}_\Omega) = \lambda$. □

Claim 2. *$\overline{T}$ and $\underline{T}$ are positively homogeneous, i.e., $\overline{T}(\lambda\xi) = \lambda\overline{T}(\xi)$ and $\underline{T}(\lambda\xi) = \lambda\underline{T}(\xi)$ for all $\xi \in \Xi$ and $\lambda \geq 0$ with $\lambda\xi \in \Xi$. Consequently, they have unique extensions to the set of all Σ-measurable real functions on Ω that preserve positive homogeneity.*

Proof. Let us show that $\overline{T}$ is positively homogeneous. The claim is trivial when $\lambda = 1$, and it follows from Claim 1 when $\lambda = 0$. Moreover, the claim for $\lambda > 1$ are implied by that for $\lambda \in (0, 1)$. As such, given that positive homogeneity is obtained for $\lambda \in (0, 1)$, we would have $\frac{1}{\rho}\overline{T}(\rho\xi) = \overline{T}(\xi)$ for an arbitrary $\rho > 1$ with $\rho\xi \in \Xi$, from which $\overline{T}(\rho\xi) = \rho\overline{T}(\xi)$. Therefore, we assume that $\lambda \in (0, 1)$ henceforth. Let $f \in \mathcal{F}$ be an act such that $\xi = u \circ f$. By $[-1, 1] \subseteq \mathrm{Im}(u)$, we can find an outcome $x_0 \in X$ such that $u(x_0) = 0$. By the fact that u is affine, we have

$$\begin{aligned}
\overline{T}(\lambda\xi) = \overline{T}(\lambda u \circ f) &= \overline{T}(u \circ (\lambda f + (1-\lambda)x_0)) \\
&= \overline{U}(\lambda f + (1-\lambda)x_0) \\
&= \lambda\overline{U}(f) + (1-\lambda)u(x_0) = \lambda\overline{U}(f) = \lambda\overline{T}(\xi).
\end{aligned}$$

where the third line follows from the fact that $\overline{U}$ is C-affine. Hence, we have shown that $\overline{T}$ is positively homogeneous.

By similar arguments, we can show that $\underline{T}$ is positively homogeneous. Finally, we claim that these functionals can be uniquely extended to $\mathcal{B}$ by preserving positive homogeneity. Indeed, for any non-zero bounded ξ, we can consider the normalized functional $\tilde{\xi} = \frac{\xi}{\|\xi\|_\infty}$, but $\tilde{\xi} \in \Xi$ since $[-1, 1] \subseteq \mathrm{Im}(u)$. To maintain positive homogeneity, we must have $\overline{T}(\xi) = \|\xi\|_\infty T(\tilde{\xi})$ and $\underline{T}(\xi) = \|\xi\|_\infty T(\tilde{\xi})$, which uniquely define the extensions. □

Let us verify several properties of the functional $\overline{T}$ and $\underline{T}$. The next sequence of claims verify monotonicity, C-additivity, and sub/super-additive.[25]

Claim 3. *$\overline{T}$ and $\underline{T}$ are C-monotonic.*

Proof. Clear from **A6** and positive homogeneity. □

Claim 4. *$\overline{T}$ and $\underline{T}$ are C-additive.*

Proof. Recall that $\overline{U}$ is C-affine, and thus, $\overline{T}(\frac{1}{2}\xi + \frac{1}{2}c\mathbf{1}_\Omega) = \frac{1}{2}\overline{T}(\xi) + \frac{1}{2}c$ trivially holds whenever $\xi \in \Xi$ and $c \in \mathrm{Im}(u)$. To generalize this observation, given any $\xi \in \mathcal{B}$ and $\lambda \in \mathbb{R}$, let $K = \max\{\|\xi\|_\infty, |\lambda|\}$.

[25] A functional $T : \mathcal{B} \to \mathbb{R}$ is *C-monotonic* if $T(\xi) \geq T(\zeta)$ whenever $\xi \geq \zeta$ and at least one of them belongs to $\mathcal{B}^c$; *C-additive* if $T(\xi + \lambda\mathbf{1}_\Omega) = T(\xi) + \lambda$ for all $\xi \in \mathcal{B}$ and $\lambda \in \mathbb{R}$; *super-additive* if $T(\xi + \zeta) \geq T(\xi) + T(\zeta)$ for all $\xi, \zeta \in \mathcal{B}$; *sub-additive* if $T(\xi + \zeta) \leq T(\xi) + T(\zeta)$ for all $\xi, \zeta \in \mathcal{B}$.

Assume $K > 0$ for nontrivial arguments. By positive homogeneity, we see that

$$\begin{aligned}\overline{T}(\xi + \lambda \mathbf{1}_\Omega) &= \overline{T}\left(2K\left(\frac{\xi}{2K} + \frac{\lambda \mathbf{1}_\Omega}{2K}\right)\right) \\ &= 2K\overline{T}\left(\frac{1}{2}\underbrace{\frac{\xi}{K}}_{\in \Xi} + \frac{1}{2}\underbrace{\frac{\lambda \mathbf{1}_\Omega}{K}}_{\in \mathrm{Im}\,(u)}\right) = 2K\left(\frac{1}{2}\overline{T}\left(\frac{\xi}{K}\right) + \frac{1}{2}\frac{\lambda}{K}\right) = \overline{T}(\xi) + \lambda,\end{aligned}$$

provided that $K > 0$. Therefore, we conclude that $\overline{T}$ is C-additive. Similarly, we can show that $\underline{T}$ is C-additive. □

Claim 5. *$\overline{T}$ is subadditive, and $\underline{T}$ is superadditive.*

Proof. As before, we assume $\xi, \zeta \in \Xi$ since the general case follows from positive homogeneity. Let $\xi = u \circ f$ and $\zeta = u \circ g$ for some $f, g \in \mathcal{F}$. We shall show that $\overline{T}(\xi + \zeta) \leq \overline{T}(\xi) + \overline{T}(\zeta)$, or equivalently, $\frac{1}{2}\overline{T}(\xi + \zeta) \leq \frac{1}{2}\overline{T}(\xi) + \frac{1}{2}\overline{T}(\zeta)$. By positive homogeneity, the left-side is equal to $\overline{T}(\frac{1}{2}\xi + \frac{1}{2}\zeta)$, which is further equal to $\overline{U}(\frac{1}{2}f + \frac{1}{2}g)$. Hence, it is enough to show that

$$\overline{U}\left(\frac{1}{2}f + \frac{1}{2}g\right) \leq \frac{1}{2}\overline{U}(f) + \frac{1}{2}\overline{U}(g), \tag{10}$$

for arbitrary $f, g \in \mathcal{F}$.

To this end, we first consider the case $\overline{U}(f) = \overline{U}(g)$. Let $x_f \in \mathcal{U}^c(f)$ and $x_g \in \mathcal{U}^c(g)$ be outcomes such that $u(x_f) = \overline{U}(f) = \overline{U}(g) = u(x_g)$. By constructions, we have $x_f \succsim f$, $x_g \succsim g$, and $x_f \sim x_g$. In particular, transitivity implies that $x_f \succsim f$ and $x_f \succsim g$, from which **A5** yields $x_f \succsim \frac{1}{2}f + \frac{1}{2}g$. Thus, $x_f \in \mathcal{U}^c(\frac{1}{2}f + \frac{1}{2}g)$. By the definition of $\overline{U}$, we have $\overline{U}(\frac{1}{2}f + \frac{1}{2}g) \leq u(x_f)$, thereby (10) is obtained.

Now, consider the case $\overline{U}(f) \neq \overline{U}(g)$. Without loss of generality, assume that $\overline{U}(f) > \overline{U}(g)$, or equivalently, $\overline{T}(\xi) > \overline{T}(\zeta)$. Let $\lambda = \overline{T}(\xi) - \overline{T}(\zeta) > 0$, and let $\tilde{\zeta} = \zeta + \lambda \mathbf{1}_\Omega$. By the C-additivity of $\overline{T}$, $\overline{T}(\tilde{\zeta}) = \overline{T}(\zeta) + \lambda = \overline{T}(\xi)$. Thus, applying the conclusion of the previous case, we get

$$\overline{T}(\xi + \tilde{\zeta}) \leq \overline{T}(\xi) + \overline{T}(\tilde{\zeta}) = \overline{T}(\xi) + \overline{T}(\zeta) + \lambda.$$

On the other hand, the C-additivity of $\overline{T}$ yields

$$\overline{T}(\xi + \tilde{\zeta}) = \overline{T}(\xi + \zeta + \lambda \mathbf{1}_\Omega) = \overline{T}(\xi + \zeta) + \lambda.$$

The above equations together imply $\overline{T}(\xi + \tilde{\zeta}) \leq \overline{T}(\xi) + \overline{T}(\zeta)$, which shows that $\overline{T}$ is subadditive. The symmetric argument proves that $\underline{T}$ is superadditive, whereas inequalities must be flipped due to the converse implication of **A5**. □

Claim 6. *$\overline{T}$ and $\underline{T}$ are continuous in $\|\cdot\|_\infty$.*

Proof. Since $\overline{T}$ is subadditive by Claim 5, it suffices to show that $\overline{T}$ is continuous at 0 (cf. Lemma 5.51 of Aliprantis and Border, 2006). Fix any $\epsilon > 0$, and consider two points $\pm 2\epsilon \mathbf{1}_\Omega \in \mathcal{B}$. Note that any ξ with $\|\xi\|_\infty < \epsilon$ is bounded by these two points. Hence, C-monotonicity implies

$$-2\epsilon = \overline{T}(-2\epsilon \mathbf{1}_\Omega) \leq \overline{T}(\xi) \leq \overline{T}(2\epsilon \mathbf{1}_\Omega) = 2\epsilon,$$

from which $|\overline{T}(\xi)| < 3\epsilon$. Thus, $\overline{T}$ is continuous at 0, and so, continuous everywhere. Moreover, we would repeat the same argument to show the continuity of $-\underline{T}$, which is subadditive since $\underline{T}$ is superadditive by Claim 5. Then, the continuity of $\underline{T}$ is implied by that of $-\underline{T}$. □

We now come to combine the functional properties obtained so far to establish the integral representations of $\overline{T}$ and $\underline{T}$. Slightly abusing the previous notation, we define the contour sets of utility acts as follows: for each $\xi \in \mathcal{B}$, Namely, for each $\xi \in \mathcal{B}$, we let

$$\mathcal{U}(\xi) = \left\{\zeta \in \mathcal{B} : \underline{T}(\zeta) \geq \overline{T}(\xi)\right\}, \quad \mathcal{U}^c(\xi) = \mathcal{U}(\xi) \cap \mathcal{B}^c,$$
$$\mathcal{L}(\xi) = \left\{\zeta \in \mathcal{B} : \underline{T}(\xi) \geq \overline{T}(\zeta)\right\}, \quad \mathcal{L}^c(\xi) = \mathcal{L}(f) \cap \mathcal{B}^c.$$

Further, let $\xi \mapsto \overline{\xi}$ and $\xi \mapsto \underline{\xi}$ be the operators from $\mathcal{B}$ to $\mathcal{B}^c$ defined by

$$\overline{\xi} = \arg\min \left\{\overline{T}(\zeta) : \zeta \in \mathcal{U}^c(\xi)\right\},$$
$$\underline{\xi} = \arg\max \left\{\underline{T}(\zeta) : \zeta \in \mathcal{L}^c(\xi)\right\}.$$

Note that these operators are well-defined.[26]

In the next claim, the observations (c) and (d) are especially important – (c) says that contour sets are invariant to the operators $\xi \mapsto \overline{\xi}, \underline{\xi}$, and then, (d) argues that contour sets of constant utility acts can be shifted along the 45° line. According to these observations, it is enough for us to characterize $\mathcal{U}(0)$ and $\mathcal{L}(0)$ to recover contour sets of arbitrary points.

Claim 7. *The following are true for any $\xi, \zeta \in \mathcal{B}$ and $\lambda \in \mathbb{R}$.*

(a) *$\mathcal{U}$ is decreasing in $\overline{T}$ in the sense that $\mathcal{U}(\xi) \subseteq \mathcal{U}(\zeta)$ whenever $\overline{T}(\xi) \geq \overline{T}(\zeta)$.*

(b) *$\mathcal{L}$ is increasing in $\underline{T}$ in the sense that $\mathcal{L}(\xi) \supseteq \mathcal{L}(\zeta)$ whenever $\underline{T}(\xi) \geq \underline{T}(\zeta)$.*

(c) *$\mathcal{U}(\xi) = \mathcal{U}(\overline{\xi})$ and $\mathcal{L}(\xi) = \mathcal{L}(\underline{\xi})$.*

(d) *$\mathcal{U}(\lambda \mathbf{1}_\Omega) = \mathcal{U}(0) + \lambda \mathbf{1}_\Omega$ and $\mathcal{L}(\lambda \mathbf{1}_\Omega) = \mathcal{L}(0) + \lambda \mathbf{1}_\Omega$.*

(e) *$\mathcal{U}(0)$ and $\mathcal{L}(0)$ are closed convex cones such that $\mathcal{B}_+ \subseteq \mathcal{U}(0)$ and $\mathcal{B}_- \subseteq \mathcal{L}(0)$.*

---

[26] To see this, note firstly that $\mathcal{U}^c(\xi)$ and $\mathcal{L}^c(\xi)$ are non-empty. Also, $\overline{T}(\lambda \mathbf{1}_\Omega) = \underline{T}(\lambda \mathbf{1}_\Omega) = \lambda$ hold by Claim 1, which means that $\overline{T}$ and $\underline{T}$ are strictly increasing and continuous on $\mathcal{B}^c$ (which can be identified with $\mathbb{R}$). Therefore, $\overline{\xi}$ and $\underline{\xi}$ are uniquely determined for every $\xi \in \mathcal{B}$.

Proof. By the definitions of contour sets, (a) and (b) are obviously true. Then, they together imply that it is enough for (c) to prove that $\overline{T}(\xi) = \overline{T}(\bar{\xi})$ and $\underline{T}(\xi) = \underline{T}(\underline{\xi})$ hold for any ξ. Note that $\overline{T}(\xi) \leq \overline{T}(\bar{\xi})$ and $\underline{T}(\xi) \geq \underline{T}(\underline{\xi})$ by the constructions of operators $\xi \mapsto \bar{\xi}, \underline{\xi}$. Suppose not, one of these inequality is strict. Since the argument is symmetric, assume that $\overline{T}(\xi) < \overline{T}(\bar{\xi})$. Then, we can pick a number $\overline{T}(\xi) < \lambda < \overline{T}(\bar{\xi})$, but then, $\overline{T}(\lambda \mathbf{1}_\Omega) = \lambda < \overline{T}(\bar{\xi})$ and $\lambda \mathbf{1}_\Omega \in \mathcal{U}^c(\xi)$ hold. This is a contradiction to the minimality of $\bar{\xi}$ within $\mathcal{U}^c(\xi)$. Hence, we must have $\overline{T}(\xi) = \overline{T}(\bar{\xi})$ and $\underline{T}(\xi) = \underline{T}(\underline{\xi})$, thereby (c) is obtained.

To see (d), we use the C-additivity of $\underline{T}$ proved in Claim 4. Observe that

$$
\begin{aligned}
\xi \in \mathcal{U}(\lambda \mathbf{1}_\Omega) &\Longleftrightarrow \underline{T}(\xi) \geq \overline{T}(\lambda \mathbf{1}_\Omega) = \underline{T}(\lambda \mathbf{1}_\Omega) \\
&\Longleftrightarrow \underline{T}(\xi - \lambda \mathbf{1}_\Omega) \geq 0 \\
&\Longleftrightarrow (\xi - \lambda \mathbf{1}_\Omega) \in \mathcal{U}(0) \\
&\Longleftrightarrow \xi \in \mathcal{U}(0) + \lambda \mathbf{1}_\Omega,
\end{aligned}
$$

from which $\mathcal{U}(\lambda \mathbf{1}_\Omega) = \mathcal{U}(0) + \lambda \mathbf{1}_\Omega$. Similarly, using the C-additivity of $\overline{T}$, we see that $\mathcal{L}(\lambda \mathbf{1}_\Omega) = \mathcal{L}(0) + \lambda \mathbf{1}_\Omega$, as required.

To see (e), recall that convexity follows from sub/super-additivity (proved in Claim 5), and closedness follows from continuity (proved in Claim 6). Moreover, by positive homogeneity (proved in Claim 2), we know that $\mathcal{U}(0)$ and $\mathcal{L}(0)$ are cones. Finally, we have $\mathcal{B}_+ \subseteq \mathcal{U}(0)$ and $\mathcal{B}_- \subseteq \mathcal{L}(0)$ by C-monotonicity observed in Claim 3. □

We characterize convex closed cones $\mathcal{U}(0)$ and $\mathcal{L}(0)$ as the intersections of supporting hyperplanes. The following proof is based on the standard separating hyperplane argument.

Claim 8. *There exist non-empty, closed, and convex sets $C^\flat, C^\sharp \subseteq \Delta(\Omega)$ such that*

$$
\mathcal{U}(0) = \left\{ \xi \in \mathcal{B} : \min_{\mu \in C^\flat} \int \xi d\mu \geq 0 \right\}, \tag{11}
$$

$$
\mathcal{L}(0) = \left\{ \xi \in \mathcal{B} : \max_{\mu \in C^\sharp} \int \xi d\mu \leq 0 \right\}. \tag{12}
$$

Proof. Let $\mathcal{B}^*$ denote the norm dual of $\mathcal{B}$. By the Riesz representation theorem (cf. Corollary 14.11 of Aliprantis and Border, 2006), $\mathcal{B}^*$ is isometrically isomorphic to the collection of all signed charges having bounded variation on (Ω, Σ), which we denote by $ba(\Omega, \Sigma)$. Now, consider any $\zeta \in \mathcal{B} \setminus \mathcal{U}(0)$. Since $\mathcal{U}(0) \cap \{\zeta\} = \emptyset$ and $\mathcal{U}(0)$ is closed and convex by Claim 7, using the separating hyperplane theorem (cf. Corollary 7.47 of Aliprantis and Border, 2006), there exists a non-zero bounded linear functional $\mu_\zeta \in \mathcal{B}^*$, which can be identified with an element of $ba(\Omega, \Sigma)$, such that

$$
\min_{\xi \in \mathcal{U}(0)} \int \xi d\mu_\zeta \equiv b > \int \zeta d\mu_\zeta.
$$

In particular, since $\mathcal{U}(0)$ is a cone that contains $\mathcal{B}_+$, μ_ζ is positive, so we can normalize it to be a probability charge. Since $0 \in \mathcal{U}(0)$, we have $b \leq 0$. Indeed, $b = 0$ holds since otherwise, there exists some $\xi \in \mathcal{U}(0)$ such that $\int \xi d\mu_\zeta < 0$, but then the left-side of the above equation can be arbitrarily small by taking $c\xi \in \mathcal{U}(0)$ with large $c > 0$. Therefore, we have shown that for each $\zeta \in \mathcal{B} \setminus \mathcal{U}(0)$, there exists a probability charge μ_ζ over (Ω, Σ) such that

$$\min_{\xi \in \mathcal{U}(0)} \int \xi d\mu_\zeta = 0 > \int \zeta d\mu_\zeta.$$

Now, let $C^\flat = \text{Cl}\ (\text{Co}\{\mu_\zeta : \zeta \in \mathcal{B} \setminus \mathcal{U}(0)\})$, which is closed and convex by construction. Note that $\min_{\mu \in C^\flat} \int \xi d\mu \geq 0$ holds for any $\xi \in \mathcal{U}(0)$, whereas we have $\int \zeta d\mu_\zeta < 0$ for any $\zeta \notin \mathcal{U}(0)$ by taking $\mu_\zeta \in C^\flat$. Therefore, $\mathcal{U}(0)$ is characterized as in (11).

Symmetrically, for any $\zeta \in \mathcal{B} \setminus \mathcal{L}(0)$, the separating hyperplane theorem yields a non-zero signed charge $\mu_\zeta \in ba(\Omega, \Sigma)$ such that

$$\max_{\xi \in \mathcal{L}(0)} \int \xi d\mu_\zeta \equiv a < \int \zeta d\mu_\zeta.$$

Again, the same argument shows that $a = 0$, and μ_ζ is positive, meaning that we can normalize it to be a probability charge. Hence, letting $C^\sharp = \text{Cl}\ (\text{Co}\{\mu_\zeta : \zeta \in \mathcal{B} \setminus \mathcal{L}(0)\})$ yields the characterization of $\mathcal{L}(0)$ given as in (12). □

Note that $\overline{\zeta} \in \mathcal{U}(\zeta)$, or equivalently $\zeta \in \mathcal{L}(\overline{\zeta})$, holds by construction. Thus, Claim 7 and 8 imply that

$$\begin{aligned} \zeta \in \mathcal{L}(\overline{\zeta}) = \mathcal{L}(0) + \overline{\zeta} \Longleftrightarrow \zeta - \overline{\zeta} \in \mathcal{L}(0) \\ \Longleftrightarrow \max_{\mu \in C^\sharp} \int (\zeta - \overline{\zeta}) d\mu \leq 0 \Longleftrightarrow \max_{\mu \in C^\sharp} \int \zeta d\mu \leq \overline{\zeta}, \end{aligned} \tag{13}$$

where $\overline{\zeta}$ is identified with the corresponding real number in the right-side. We claim that (13) must be tight. Suppose not, the inequality is strict, and let $\zeta_0 \in \mathcal{B}^c$ be a constant utility act such that $\max_{\mu \in C^\sharp} \int \zeta d\mu < \zeta_0 < \overline{\zeta}$. A contradiction would be derived if we can show that $\zeta \in \mathcal{L}(\zeta_0)$. As such, by Claim 8, this is equivalent to

$$\begin{aligned} \zeta \in \mathcal{L}(\zeta_0) = \mathcal{L}(0) + \zeta_0 \Longleftrightarrow \zeta - \zeta_0 \in \mathcal{L}(0) \\ \Longleftrightarrow \max_{\mu \in C^\sharp} \int (\zeta - \zeta_0) d\mu \leq 0 \Longleftrightarrow \max_{\mu \in C^\sharp} \int \zeta d\mu \leq \zeta_0, \end{aligned}$$

where the last assertion is true by the assumption, hence, we encounter a contradiction. Therefore, any $\zeta \in \mathcal{B}$ satisfies the following identity:

$$\overline{\zeta} = \max_{\mu \in C^\sharp} \int \zeta d\mu, \tag{14}$$

where the constant function $\overline{\zeta}$ is identified with the real number corresponding to its anonymous coordinate.[27]

Now, fix any $f, g \in \mathcal{F}$ with $f \neq g$, and denote their utility acts by $\xi = u \circ f$ and $\zeta = u \circ g$. Combining Claim 7 and 8, we see that

$$\begin{aligned} f \succsim g &\Longleftrightarrow \underline{U}(f) \geq \overline{U}(g) \\ &\Longleftrightarrow \underline{T}(\xi) \geq \overline{T}(\zeta) \\ &\Longleftrightarrow \xi \in \mathcal{U}(\zeta) = \mathcal{U}(\overline{\zeta}) = \mathcal{U}(0) + \overline{\zeta} \\ &\Longleftrightarrow \xi - \overline{\zeta} \in \mathcal{U}(0) \\ &\Longleftrightarrow \min_{\mu \in C^\flat} \int (\xi - \overline{\zeta}) d\mu \geq 0 \\ &\Longleftrightarrow \min_{\mu \in C^\flat} \int \xi d\mu \geq \overline{\zeta}. \end{aligned} \tag{15}$$

Substituting (14) into (15), we obtain

$$f \succsim g \Longleftrightarrow \min_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \max_{\mu \in C^\sharp} \int (u \circ g) d\mu, \tag{16}$$

for any $f \neq g$, so that the desired representation is obtained.

Note that $C^\sharp$ and $C^\flat$ are nonempty, closed, and convex by construction. Our remaining task is to show that they are not disjoint. To this end, suppose by contradiction that $C^\sharp \cap C^\flat = \emptyset$. Since these sets are weak-* compact subsets of $\mathcal{B}^*$, the strong separating hyperplane theorem yields a bounded linear functional $\xi^{**} \in \mathcal{B}^{**}$ such that $\min_{\mu \in C^\flat} \langle \xi^{**}, \mu \rangle > \max_{\mu \in C^\sharp} \langle \xi^{**}, \mu \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product. In particular, by the normalization of u, there exists an act $f \in \mathcal{F}$ such that

$$v_1 \equiv \min_{\mu \in C^\flat} \int (u \circ f) d\mu > \max_{\mu \in C^\sharp} \int (u \circ f) d\mu \equiv v_2. \tag{17}$$

Now, let $x, x' \in X$ be any outcomes with $u(x) > u(x')$ such that $u(x), u(x') \in (v_2, v_1)$. By the representation (14), we have $f \sim x$ and $f \sim x'$, but $x \succ x'$, meaning that $\succsim$ violates transitivity. Therefore, $C^\sharp \cap C^\flat \neq \emptyset$.[28]

[27] The symmetric identity, $\underline{\zeta} = \inf_{\mu \in C^\flat} \int \zeta d\mu$, can also be verified though we do not use it in the proof.

[28] Hence, we have shown that the transitivity of $\succsim$ implies that (17) does *not* hold for all acts, which in turn implies that $C^\sharp \cap C^\flat \neq \emptyset$. In the supplementary appendix of this paper, we further show that these conditions are equivalent; see Appendix D.1.2. Hence, in other words, the non-emptiness of belief sets ensures transitivity, but it has no role beyond

Step 2: Uniqueness of belief sets.

We shall prove that $C^\sharp$ and $C^\flat$ are uniquely determined. Given a utility function $u : X \to \mathbb{R}$, suppose by contradiction that there exist two different pairs of belief sets, say $(C^\sharp, C^\flat)$ and $(D^\sharp, D^\flat)$ that represent the same preference $\succsim$. Without loss of generality, assume that there exists some $\mu^\sharp \in C^\sharp \setminus D^\sharp$. Applying the separating hyperplane theorem to $\{\mu^\sharp\}$ and $D^\sharp$, we obtain a non-zero bounded linear functional $\xi^{**} \in \mathcal{B}^{**}$ and a scalar $\lambda \in \mathbb{R}$ such that

$$\max_{\mu \in D^\sharp} \langle \xi^{**}, \mu \rangle \leq \lambda < \langle \xi^{**}, \mu^\sharp \rangle. \tag{18}$$

In particular, we can let $\xi^{**} \equiv \xi \in \mathcal{B}$. If we let $\lambda \mathbf{1}_\Omega \in \mathcal{B}$ be a constant function, then

$$\min_{\mu \in C^\flat} \langle \lambda \mathbf{1}_\Omega, \mu \rangle = \min_{\mu \in D^\flat} \langle \lambda \mathbf{1}_\Omega, \mu \rangle = \lambda. \tag{19}$$

Let $K = \max\{\|\xi\|_\infty, |\lambda|\}$, and observe that $K > 0$. Combining (18) and (19), it follows that

$$\frac{\lambda}{K} = \min_{\mu \in D^\flat} \left\langle \frac{\lambda}{K} \mathbf{1}_\Omega, \mu \right\rangle \geq \max_{\mu \in D^\sharp} \left\langle \frac{\xi}{K}, \mu \right\rangle, \tag{20}$$

$$\frac{\lambda}{K} = \min_{\mu \in C^\flat} \left\langle \frac{\lambda}{K} \mathbf{1}_\Omega, \mu \right\rangle < \left\langle \frac{\xi}{K}, \mu^\sharp \right\rangle \leq \max_{\mu \in C^\sharp} \left\langle \frac{\xi}{K}, \mu \right\rangle, \tag{21}$$

while we have $\frac{\xi}{K}, \frac{\lambda \mathbf{1}_\Omega}{K} \in [-1,1]^\Omega \cap \mathcal{B} \subseteq \Xi$. Hence, there exist $f \in \mathcal{F}$ and $x \in X$ for which $u \circ f = \frac{\xi}{K}$ and $u(x) = \frac{\lambda}{K}$, from which $x \succsim f$ according to (20). However, (21) implies that $x \not\succsim f$, a contradiction. *Q.E.D.*

Appendix C Proofs for Sections 4 and 5

We adopt the same notation as the previous section. Let $\mathcal{B}(\Omega, \Sigma)$, or simply $\mathcal{B}$, denote the set of all bounded Σ-measurable real functions on Ω, endowed with the sup norm $\|\cdot\|_\infty$. Denote by $\mathcal{B}^*$ the norm dual of $\mathcal{B}$, and by $\mathcal{B}^{**}$ the double dual. We sometimes use inner product notation: $\langle \xi^*, \xi \rangle$ stands for a functional $\xi^* \in \mathcal{B}^*$ acting on $\xi \in \mathcal{B}$ etc.

C.1 Proof of Proposition 1

We only prove the first equivalence since the second can be symmetrically discussed, and the third is a direct corollary to these statements. That is, we want to show that $C^\sharp \supseteq C^\flat$ if and only if **A9 (a)** is satisfied.

Necessity. Suppose that $C^\sharp \supseteq C^\flat$. Take any $f, g \in \mathcal{F}$ with $f \succsim g$ and $f \overset{\alpha}{\smile} g$. By the definition of

that.

$f \overset{\alpha}{\smile} g$, there exists $\lambda \in \mathbb{R}$ for which

$$\begin{aligned}
&\alpha u \circ f + (1-\alpha) u \circ g = \lambda \mathbf{1}_\Omega \\
&\Longleftrightarrow u \circ g = \frac{\lambda}{1-\alpha}\mathbf{1}_\Omega - \frac{\alpha}{1-\alpha} u \circ f.
\end{aligned} \tag{22}$$

Then, we see that

$$\begin{aligned}
&\max_{\mu \in C^\sharp} \int \big(u \circ (\alpha f + (1-\alpha) g)\big) d\mu = \max_{\mu \in C^\sharp} \int \lambda \mathbf{1}_\Omega d\mu = \lambda \\
&= \alpha \min_{\mu \in C^\flat} \int (u \circ f) d\mu - \alpha \min_{\mu \in C^\flat} \int (u \circ f) d\mu + \lambda \\
&= \alpha \min_{\mu \in C^\flat} \int (u \circ f) d\mu + (1-\alpha) \max_{\mu \in C^\flat} \int \left(\frac{\lambda}{1-\alpha}\mathbf{1}_\Omega - \frac{\alpha}{1-\alpha} u \circ f \right) d\mu \\
&= \alpha \min_{\mu \in C^\flat} \int (u \circ f) d\mu + (1-\alpha) \max_{\mu \in C^\flat} \int (u \circ g) d\mu \\
&\leq \alpha \min_{\mu \in C^\flat} \int (u \circ f) d\mu + (1-\alpha) \max_{\mu \in C^\sharp} \int (u \circ g) d\mu \\
&\leq \alpha \min_{\mu \in C^\flat} \int (u \circ f) d\mu + (1-\alpha) \min_{\mu \in C^\flat} \int (u \circ f) d\mu \\
&= \min_{\mu \in C^\flat} \int (u \circ f) d\mu,
\end{aligned}$$

where the forth line follows from (22), the fifth line uses the assumption $C^\sharp \supseteq C^\flat$, and the sixth line is due to $f \succsim g$. Therefore, $f \succsim \alpha f + (1-\alpha) g$, from which **A9 (a)** is satisfied.

Sufficiency. suppose that there exists some $\mu^\flat \in C^\flat \setminus C^\sharp$. As usual, assume that $[-1, 1] \subseteq \mathrm{Im}(u)$. The separating hyperplane theorem yields a non-zero bounded linear functional $\xi^{**} \in \mathcal{B}^{**}$ and a scalar $c \in \mathbb{R}$ such that $\langle \xi^{**}, \mu^\flat \rangle < c < \langle \xi^{**}, \mu^\sharp \rangle$ for all $\mu^\sharp \in C^\sharp$. Without loss of generality, we let $\xi = \xi^{**}$ be in $\mathcal{B}$. By the compactness of $C^\sharp$, we have

$$\min_{\mu \in C^\flat} \langle \xi, \mu \rangle < \min_{\mu \in C^\sharp} \langle \xi, \mu \rangle. \tag{23}$$

Define $\gamma = \max\{\|\xi\|_\infty, \gamma_0\}$, where $\gamma_0 = |\min_{\mu \in C^\sharp} \langle \xi, \mu \rangle| + |\min_{\mu \in C^\flat} \langle \xi, \mu \rangle|$, and observe that $\gamma, \gamma_0 > 0$. Furthermore, we set $\zeta = \frac{\xi}{\gamma}$ and $\lambda = \min_{\mu \in C^\sharp} \langle \zeta, \mu \rangle + \min_{\mu \in C^\flat} \langle \zeta, \mu \rangle$. By construction, $\zeta \in [-1, 1]^\Omega$ holds, and (23) gives rise to

$$\min_{\mu \in C^\flat} \langle \zeta, \mu \rangle < \min_{\mu \in C^\sharp} \langle \zeta, \mu \rangle. \tag{24}$$

Also, observe that

$$|\lambda| \le \left|\min_{\mu\in C^\sharp}\langle\zeta,\mu\rangle\right| + \left|\min_{\mu\in C^\flat}\langle\zeta,\mu\rangle\right| = \frac{\gamma_0}{\gamma} \le 1,$$

from which $\lambda \in [-1,1]$. Hence, we can find some acts $f, h \in \mathcal{F}$ such that $u \circ f = \frac{1}{2}\zeta$ and $u \circ h = -\zeta$, and an outcome $x \in X$ such that $u(x) = \lambda$.

Now, let $g = \frac{1}{2}h + \frac{1}{2}x$. We shall show that **A9 (a)** is violated by f, g and their fair mixture $\frac{1}{2}f + \frac{1}{2}g$. To this end, first observe that

$$u \circ \left(\frac{1}{2}f + \frac{1}{2}g\right) = \frac{1}{2}u\circ f + \frac{1}{4}u\circ h + \frac{1}{4}u(x)\mathbf{1}_\Omega = \frac{1}{4}\zeta - \frac{1}{4}\zeta + \frac{1}{4}\lambda\mathbf{1}_\Omega = \frac{1}{4}\lambda\mathbf{1}_\Omega, \tag{25}$$

meaning that $f \overset{1/2}{\smile} g$. Moreover, we see that

$$\begin{aligned}\max_{\mu\in C^\sharp}\int(u\circ g)d\mu &= \frac{1}{2}\max_{\mu\in C^\sharp}\int(u\circ h)d\mu + \frac{1}{2}u(x) = \frac{1}{2}\max_{\mu\in C^\sharp}\langle-\zeta,\mu\rangle + \frac{1}{2}\lambda \\ &= -\frac{1}{2}\min_{\mu\in C^\sharp}\langle\zeta,\mu\rangle + \frac{1}{2}\left(\min_{\mu\in C^\sharp}\langle\zeta,\mu\rangle + \min_{\mu\in C^\flat}\langle\zeta,\mu\rangle\right) \\ &= \frac{1}{2}\min_{\mu\in C^\flat}\langle\zeta,\mu\rangle = \min_{\mu\in C^\flat}\int(u\circ f)d\mu,\end{aligned}$$

from which $f \succsim g$. To falsify **A9 (a)**, it remains to show that

$$\min_{\mu\in C^\flat}\int(u\circ f)d\mu < \max_{\mu\in C^\sharp}\int u\circ\left(\frac{1}{2}f + \frac{1}{2}g\right)d\mu,$$

which is, by (25) and the constructions of f and λ, equivalent to the following inequality:

$$\min_{\mu\in C^\flat}\langle\zeta,\mu\rangle < \frac{1}{2}\underbrace{\left(\min_{\mu\in C^\sharp}\langle\zeta,\mu\rangle + \min_{\mu\in C^\flat}\langle\zeta,\mu\rangle\right)}_{\equiv\lambda}. \tag{26}$$

Indeed, (24) implies that (26) is true, and thus, we are done. *Q.E.D.*

C.2 Proof of Proposition 2

Suppose that (ii) is true. We can let $u \equiv u_1 = u_2$ without loss of generality. Clearly, $\succsim_1|_{\mathcal{F}^c} = \succsim_2|_{\mathcal{F}^c}$. Fix any $f, g \in \mathcal{F}$ with $f \succsim_2 g$. If $f = g$, then both $f \sim_1 g$ and $f \sim_2 g$ hold by reflexivity, so there is nothing to prove. Otherwise, it holds that

$$\min_{\mu\in C_1^\flat}\int(u\circ f)d\mu \ge \min_{\mu\in C_2^\flat}\int(u\circ f)d\mu \ge \max_{\mu\in C_2^\sharp}\int(u\circ g)d\mu \ge \max_{\mu\in C_1^\sharp}\int(u\circ g)d\mu,$$

Table 1: Acts in Proposition 3

	ω	$\Omega \setminus \{\omega\}$
$u \circ f$	γ	α
$u \circ g$	β	δ

where the first and third inequalities follow from $C_1^\flat \subseteq C_2^\flat$ and $C_1^\sharp \subseteq C_2^\sharp$, respectively. Hence, we obtain $f \succsim_1 g$, from which $\succsim_1$ is a compatible extension of $\succsim_2$.

Conversely, suppose that (ii) is false. If u_1 is not a positive affine transformation of u_2, the two preferences differ on $\mathcal{F}^c$, from which one cannot be a compatible extension of another. Thus, we can let $u \equiv u_1 = u_2$. Since (ii) fails, we have either $\mu \in C_1^\sharp \setminus C_2^\sharp$ or $\mu \in C_1^\flat \setminus C_2^\flat$. Assuming the former case, the separating hyperplane theorem yields $\xi^{**} \in \mathcal{B}^{**}$ such that

$$\langle \xi^{**}, \mu \rangle > c \equiv \max_{\mu_2 \in C_2^\sharp} \langle \xi^{**}, \mu_2 \rangle. \tag{27}$$

After normalization, we can let $\xi^{**} = u \circ f$ for some $f \in \mathcal{F}$. Now, let $x \in X$ be an outcome such that $u(x) = c$. By (27), it follows that $x\mathbf{1}_\Omega \succsim_2 f$. On the other hand, since $\mu \in C_1^\sharp$,

$$\max_{\mu_1 \in C_1^\sharp} \int (u \circ f) d\mu_1 > u(x),$$

whereas $f \neq x$ since f must not be constant to maintain (27). Thus, we have $x \not\succsim_1 f$, from which $\succsim_1$ is not a compatible extension of $\succsim_2$. The case in which $\mu \in C_1^\flat \setminus C_2^\flat$ is similarly discussed. *Q.E.D.*

C.3 Proof of Proposition 3

Clearly, the representation in this proposition is the special case of Theorem 1, corresponding to the case of when $C^\sharp = C^\flat = \Delta(\Omega)$. In particular, it is easy to see that the representation satisfies **A10**. To prove the sufficiency, suppose that $\succsim$ satisfies all the listed axioms, and thus, admit an IPOP representation by some $(C^\sharp, C^\flat, u)$. Let $[0,1] \subseteq \mathrm{Im}(u)$ without loss of generality. It is enough to show that $c^\sharp = c^\flat = 1$, where $c^\natural = \max_{\mu \in C^\natural} \mu(\{\omega\})$ for each $\omega \in \Omega$ and $\natural \in \{\sharp, \flat\}$. To this end, let $\alpha > \beta > \gamma > \delta$ be arbitrary numbers in $[0,1]$, and consider acts $f, g \in \mathcal{F}$ which pay the utility values, in each event $\{\omega\}$ and $\Omega \setminus \{\omega\}$, as being summarized in Table 1. Note that f does not dominate g, and so, **A10** implies that $f \not\succsim g$. That is, we must have

$$\min_{\mu \in C^\flat} \int (u \circ f) d\mu = c^\flat \gamma + (1 - c^\flat)\alpha$$
$$< c^\sharp \beta + (1 - c^\sharp)\delta = \max_{\mu \in C^\sharp} \int (u \circ g) d\mu.$$

which is satisfied for *all* $\alpha > \beta > \gamma > \delta$ if and only if $c^\flat = c^\sharp = 1$. *Q.E.D.*

C.4 Proof of Proposition 4

Clearly, $\mathcal{P}_{\text{MEU}} \cap \mathcal{P}_{\text{Bewley}} = \mathcal{P}_{\text{MEU}} \cap \mathcal{P}_{\text{IPOP}} = \mathcal{P}_{\text{SEU}}$ because IPOP and Bewley preferences would be complete if and only if the associated prior set is singleton. Note that any $\succsim_{\text{Bewley}} \in \mathcal{P}_{\text{Bewley}}$ can be extended to $\succsim^* \in \mathcal{P}_{\text{SEU}} \subseteq \mathcal{P}_{\text{MEU}}$ by letting $\succsim^*$ be a SEU preference defined by the same utility function u as $\succsim_{\text{Bewley}}$, and an arbitrary μ in the belief set of $\succsim_{\text{Bewley}}$. Thus, the crucial part would be that any $\succsim_{\text{MEU}} \in \mathcal{P}_{\text{MEU}}$ has some $\succsim_{\text{Bewley}} \in \mathcal{P}_{\text{Bewley}}$ for which $\succsim_{\text{MEU}}$ is a compatible extension of $\succsim_{\text{Bewley}}$. Indeed, this claim follows from the combination of Definition 3 and Proposition 4 and 5 in Ghirardato et al. (2004), thereby we obtain (i). Then, notice that a more-conservative relation is a transitive order over nonempty classes of preferences. Hence, (ii) will follow from the combination of (i) and (iii).

Our remaining task is to prove (iii). We first show that any $\succsim \in \mathcal{P}_{\text{IPOP}}$ satisfies independence if and only if $C^\sharp = C^\flat = \{\mu_0\}$ for some $\mu_0 \in \Delta(\Omega)$, i.e., $\succsim \in \mathcal{P}_{\text{SEU}}$. The if direction is trivial, so let us consider the only if direction. Pick any $\mu_0 \in C^\sharp \cap C^\flat$, and suppose that there exists some $\mu_1 \in C^\sharp \setminus \{\mu_0\}$. By the separating hyperplane theorem, there is an act $f \in \mathcal{F}$ such that

$$\max_{\mu \in C^\sharp} \int (u \circ f) d\mu \geq \int (u \circ f) d\mu_1 > \int (u \circ f) d\mu_0 \geq \min_{\mu \in C^\flat} \int (u \circ f) d\mu.$$

Let $x \in X$ be an outcome such that $u(x) = \min_{\mu \in C^\flat} \int (u \circ f) d\mu$. While $f \succsim x$ by the construction, we see for an arbitrary $\lambda \in (0,1)$ that

$$\begin{aligned}
&\min_{\mu \in C^\flat} \int \big(u \circ (\lambda f + (1-\lambda) f)\big) d\mu \\
&\leq \lambda \min_{\mu \in C^\flat} \int (u \circ f) d\mu + (1-\lambda) \min_{\mu \in C^\flat} \int (u \circ f) d\mu \\
&< \lambda u(x) + (1-\lambda) \max_{\mu \in C^\sharp} \int (u \circ f) d\mu = \max_{\mu \in C^\sharp} \int \big(u \circ (\lambda x + (1-\lambda) f)\big) d\mu,
\end{aligned}$$

from which $\lambda f + (1-\lambda) f \not\succsim \lambda x + (1-\lambda) f$. Therefore, $\succsim$ violates independence. Consequently, since Bewley preferences satisfy independence, we confirm that $\mathcal{P}_{\text{Bewley}} \cap \mathcal{P}_{\text{IPOP}} = \mathcal{P}_{\text{SEU}}$.

Next, we shall show that $\mathcal{P}_{\text{IPOP}}$ is more conservative than $\mathcal{P}_{\text{Bewley}}$. Again, note that any $\succsim_{\text{IPOP}} \in \mathcal{P}_{\text{IPOP}}$ can be extended to $\succsim^* \in \mathcal{P}_{\text{Bewley}}$, as any SEU preference $\succsim^*$ defined by u and an arbitrary $\mu \in C^\sharp \cap C^\flat$ does work. Conversely, fix any $\succsim_{\text{Bewley}} \in \mathcal{P}_{\text{Bewley}}$ that is represented by (u, C). Then, we define $\succsim_{\text{IPOP}} \in \mathcal{P}_{\text{IPOP}}$ by using the same utility function u, and setting $C = C^\sharp = C^\flat$. Clearly, $\succsim_{\text{Bewley}} |_{\mathcal{F}^c} = \succsim_{\text{IPOP}} |_{\mathcal{F}^c}$. Moreover, it is easy to see that $f \succsim_{\text{IPOP}} g$ whenever $f \succsim_{\text{Bewley}} g$, thereby $\succsim_{\text{Bewley}}$ is a compatible extension of $\succsim_{\text{IPOP}}$. Hence, $\mathcal{P}_{\text{IPOP}}$ is more conservative than $\mathcal{P}_{\text{Bewley}}$. *Q.E.D.*

C.5 Proof of Proposition 5

Let $\succsim^*$ be a C-continuous completion of $\succsim$. Given an act $f \in \mathcal{F}$, consider the nonempty sets $A = \{x \in \mathcal{F}^c : x \succsim^* f\}$ and $B = \{x \in \mathcal{F}^c : f \succsim^* x\}$, which are closed since $\succsim^*$ is C-continuous. Moreover, $A \cup B = X$ since $\succsim^*$ is complete, so the connectedness of $\mathcal{F}^c \simeq X$ implies that $A \cap B \neq \emptyset$. Then, we fix an arbitrary $x_f \in A \cap B$ and set $I(f) = u(x_f)$. Note that $I(f)$ is independent of how x_f is chosen because $f \sim^* x$ for every $x \in A \cap B$.

Let us show that $I(f)$ takes the form as in the statement. Since $\succsim^*$ is a compatible extension, neither $f \succ x_f$ nor $x_f \succ f$ holds to assure that $f \sim^* x_f$. Note that $f \not\succ x_f$ gives rise to either $f \not\succsim x_f$ or $x_f \succsim f$. Similarly, $x_f \not\succ f$ gives rise to either $f \succsim x_f$ or $x_f \not\succsim f$. Hence, if $f \sim^* x_f$, then f and x must be either indifferent or incomparable with respect to $\succsim$. In the former case, the minimum and maximum expected utilities of f are both equal to $u(x_f)$, and thus, $I(f)$ takes the form (4) for an arbitrary $\alpha(f) \in (0,1)$. In the latter case, we see that

$$\max_{\mu \in C^\sharp} \int (u \circ f) d\mu > u(x_f) > \min_{\mu \in C^\flat} \int (u \circ f) d\mu,$$

from which we can find an $\alpha(f) \in (0,1)$ that provides (4).

Now, we shall show that $I(\cdot)$ represents $\succsim^*$. Assume that $I(f) \succsim I(g)$. By construction, there exist $x_f \sim^* f$ and $x_g \sim^* g$ such that $u(x_f) = I(f) \geq I(g) = u(x_g)$, meaning that $x_f \succsim x_g$. In particular, this leads to $x_f \succsim^* x_g$, from which transitivity dictates $f \succsim^* g$. Furthermore, when $I(f) > I(g)$, we must have $f \sim^* x_f \succ^* x_g \sim^* g$, thereby $f \succ^* g$. Therefore, $f \succsim^* g$ if and only if $I(f) \geq I(g)$, as desired.

As to the converse direction, suppose that $\succsim^*$ is represented by $I : \mathcal{F} \to \mathbb{R}$ as in (4), where $\alpha : \mathcal{F} \to [0,1]$ is an arbitrary function. Clearly, $\succsim^*$ is a weak order, and it is C-continuous by the fact that $I(x) = u(x)$ for every $x \in \mathcal{F}^c$. Note that this implies that $\succsim|_{\mathcal{F}^c} = \succsim^*|_{\mathcal{F}^c}$. Now, take any $f, g \in \mathcal{F}$ with $f \succsim g$. If $f = g$ then $f \succsim^* g$, as $\succsim^*$ is complete. If $f \neq g$, by the transitivity of $\succsim$, we have

$$I(f) \geq \min_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \max_{\mu \in C^\sharp} \int (u \circ f) d\mu \geq I(g),$$

from which $f \succsim^* g$. Therefore, $\succsim^*$ is a compatible extension of $\succsim$. *Q.E.D.*

C.6 Proof of Corollary 1

It is easy to check the if part. For the only if part, assume that $(\succsim, \succsim^*)$ jointly satisfies caution. In view of proposition 5, we have to show that caution dictates $\alpha(f) = 1$ for all $f \in \mathcal{F}$. Clearly, we can set α in that way for any f whose expected utility does not depend on $\mu \in C$. Suppose not, there exists f such that $\alpha(f) < 1$ and $\min_{\mu \in C} \int (u \circ f) d\mu < \max_{\mu \in C} \int (u \circ f) d\mu$. As such, if $x \in \mathcal{F}^c$ is picked so that $\min_{\mu \in C} \int (u \circ f) d\mu < u(x) < \max_{\mu \in C} \int (u \circ f) d\mu$, then we have $f \not\succsim x$. However, by letting $u(x)$ be close enough to $\min_{\mu \in C} \int (u \circ f) d\mu$, we have $f \succ^* x$, which causes a contradiction to caution.

Hence, $\alpha(f) = 1$. The case of maximax is similarly discussed. *Q.E.D.*

C.7 Proof of Proposition 6

Recall that $U(\cdot, \mu)$ is maximized at v for any μ, and so is $\min_{\mu \in C^\flat} U(\cdot, \mu)$. Hence, any bid $b \neq v$ is dominated by some other b' if and only if b is dominated by v. Similarly, note that $\max_{\mu \in C^\flat} U(\cdot, \mu)$ is maximized at v so that $b \notin B^*(C^\sharp, C^\flat)$ for any $b \neq v$ whenever truth-telling is transparent. Hence, we have $B^*(C^\sharp, C^\flat) \subseteq \{v\}$ if truth-telling is transparent. Moreover, if Condition 2 is satisfied v is not dominated by any $b \neq v$, so we have $B^*(C^\sharp, C^\flat) = \{v\}$.

Now, suppose that truth-telling is not transparent, so the left-side is strictly greater than the right side in (5). This assures that $v \in B^*(C^\sharp, C^\flat)$, and thus, we have $b \in B^*(C^\sharp, C^\flat)$ if and only if $\overline{U}(b) > \underline{U}(v)$, where

$$\overline{U}(b) \equiv \max_{\mu \in C^\sharp} \underbrace{\int_0^b u(1, \omega) d\mu(\omega)}_{\equiv U(b,\mu)} \quad \text{and} \quad \underline{U}(b) \equiv \min_{\mu \in C^\flat} \int_0^v u(1, \omega) d\mu(\omega). \tag{28}$$

Thus, to see that $B^*(C^\sharp, C^\flat)$ takes an interval form, it suffices to show that $\overline{U}$ is weakly increasing on $[0, v]$, and it is weakly decreasing on $[v, \bar{b}]$. Indeed, it is straightforward to show $U(\cdot, \mu)$ has these properties for every $\mu \in C^\sharp$, and thus, so does $\overline{U}$ defined as the maximum.[29] Finally, let us show that $B^*(C^\sharp, C^\flat)$ is open (relative to $[0, \bar{b}]$). Note that $B^*(C^\sharp, C^\flat)$ is the inverse image of an open set under $\overline{U}$, so that it is enough to prove that $\overline{U}$ is continuous. To this end, fix any $\mu \in C^\sharp$. By Condition 1, we know $\mu[0, b]$ is continuous in b. Furthermore, observe that $\nu[0, b] = \int_0^b u(1, \omega) d\mu(\omega)$ is continuous in b, as ν is obtained by transforming μ with the continuous Radon-Nikodym derivative $\frac{d\nu}{d\mu}(\omega) = u(1, \omega)$. Hence, each $U(\cdot, \mu)$ is continuous, thereby so is $\overline{U}$. *Q.E.D.*

[29] More formally, recall that each $\mu \in C^\sharp$ has a probability density function ϕ by Condition 1, so that we can write $U(b, \mu) = \int_0^b u(1, \omega)\phi(\omega) d\omega$. By the Leibniz rule, it follows that $\frac{\partial U(b,\mu)}{\partial b} = u(1, b)\phi(b)$, which is weakly positive (resp. negative) if $b \leq v$ (resp. $b \geq v$) by the assumptions on u. Moreover, these properties are inherited to $\overline{U}$ because monotonicity is preserved by the max operator.

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Supplementary Material to "An Axiomatic Approach to Failures in Contingent Reasoning"

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Appendix D Additional Results

D.1 Weaker Transitivity and Disjointness of Belief Sets

Recall that we have allowed for the possibility that the DM's optimism and pessimism possess different belief sets. As such, their non-emptiness is the only essential restriction assumed in Theorem 1. We show that this restriction guarantees the transitivity of $\succsim$, but it has no role more than that. We also present other equivalent restatements of transitivity that have natural interpretations in light of the DM's rationality.

To this end, we consider the following alternative of **A1** that replaces transitivity with C-transitivity, which applies only when the middle is a constant act.[30]

B1. $\succsim$ is a non-degenerate and reflexive order such that for every $f \in \mathcal{F}$ there exist some $x, y \in \mathcal{F}^c$ for which $x \succsim f$ and $f \succsim y$. Moreover, $\succsim$ is *C-transitive*; for any $f, g \in \mathcal{F}$ and $x \in \mathcal{F}^c$, if $f \succsim x$ and $x \succsim g$, then $f \succsim g$.

D.1.1 The Role of Transitivity in General Representations

After replacing **A1** by **B1**, we continue to have our previous representation results with only minor modifications. For general representations, our Lemma 1 is restored by deleting the condition $\overline{U} \geq \underline{U}$ from its statement. The proof is essentially the same as before, so omitted.

Lemma 4. *A preference relation $\succsim$ satisfies* **B1**, **A2**, **A3**, *and* **A8** *if and only if there exist non-constant functions $\overline{U}, \underline{U} : \mathcal{F} \to \mathbb{R}$ such that*

(i) $\overline{U}|_{\mathcal{F}^c} = \underline{U}|_{\mathcal{F}^c}$ *holds, and the restriction is continuous on $\mathcal{F}^c \simeq X$;*

(ii) $\mathrm{Im}(\overline{U}|_{\mathcal{F}^c}) = \mathrm{Im}(\overline{U}) = \mathrm{Im}(\underline{U}) = \mathrm{Im}(\underline{U}|_{\mathcal{F}^c})$*; and*

(iii) $f \succsim g$ *if and only if* $\underline{U}(f) \geq \overline{U}(g)$ *or* $f = g$.

Moreover, a pair of functions $\overline{V}, \underline{V} : \mathcal{F} \to \mathbb{R}$ satisfies (i)-(iii) for the same preference relation $\succsim$ if and only if there exists a continuous and strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\overline{V} = \phi \circ \overline{U}$ and $\underline{V} = \phi \circ \underline{U}$.

[30] In this regard, C-transitivity is viewed as the reverse of C-calibration, and thus, it should be thought of as being more or less "minimal" in representation results. As such, we have essentially the same results as before both for general and expected utility representations.

The next result shows that the omitted condition $\overline{U} \geq \underline{U}$ is equivalent to the transitivity of $\succsim$. In that sense, transitivity guarantees that the optimism inside the DM is actually more "optimistic" than her pessimism. Yet another interpretation is available. (iii) of the below lemma says that the optimism and pessimism rank a pair of acts in the same order whenever the DM is decisive for the pair. Putting differently, the different selves of the DM share the same ordinal preferences on the domain on which the DM confirms solid rankings.

Lemma 5. *Suppose that $\succsim$ satisfies* **B1***,* **A2***,* **A3***, and* **A8***. Let $\overline{U}$ and $\underline{U}$ be arbitrary functions that represent $\succsim$ in the way of Lemma 4. The following conditions are equivalent:*

(i) *$\succsim$ satisfies transitivity;*

(ii) *$\overline{U}(f) \geq \underline{U}(f)$ for all $f \in \mathcal{F}$; and*

(iii) *$(\overline{U}(f) - \overline{U}(g))(\underline{U}(f) - \underline{U}(g)) \geq 0$ whenever f and g are comparable.*

Proof. It is easy to see (ii) $\Rightarrow$ (i). Let us show (i) $\Rightarrow$ (ii). Suppose not, there exists some $f \in \mathcal{F}$ for which $\overline{U}(f) < \underline{U}(f)$. Take outcomes $x, y \in X$ such that $u(x) = \overline{U}(f)$ and $u(y) = \underline{U}(f)$, where u stands for the common restriction of $\overline{U}$ and $\underline{U}$ on $\mathcal{F}^c$. Then, we have $x \succsim f$ and $f \succsim y$, so that transitivity yields $x \succsim y$. However, since $\overline{U}(f) < \underline{U}(f)$, we have $\overline{U}(x) < \underline{U}(y)$ and $\underline{U}(x) < \overline{U}(y)$, which leads to $y \succ x$, a contradiction.

Let us show that (iii) is equivalent to (ii). Note that (iii) is trivially satisfied when $f = g$. So, take any $f, g \in \mathcal{F}$ such that $f \succsim g$ and $f \neq g$, so $\underline{U}(f) \geq \overline{U}(g)$. Assuming that (ii) is true, we have $\overline{U}(f) \geq \underline{U}(f) \geq \overline{U}(g) \geq \underline{U}(g)$, from which (iii) is obtained. Conversely, suppose that (ii) is violated, i.e., there exists $f \in \mathcal{F}$ for which $\overline{U}(f) < \underline{U}(f)$. Given any number $c \in (\overline{U}(f), \underline{U}(f))$, let $z \in X$ be an outcome for which $\overline{U}(z) = \underline{U}(z) = c$. While $z \succsim f$ holds, we have

$$\underbrace{\left(\overline{U}(f) - \overline{U}(z)\right)}_{<0} \underbrace{\left(\underline{U}(f) - \underline{U}(z)\right)}_{>0} < 0,$$

from which (iii) is violated. Therefore, we have established all the desired equivalence. $\square$

D.1.2 The Role of Transitivity in Expected Utility Representations

After replacing **A1** by **B1**, an alternative result is established for expected utility representations. In the below theorem, only the difference from Theorem 1 is that the condition $C^\sharp \cap C^\flat \neq \emptyset$ is deleted. The proof is essentially the same, so omitted.

Theorem 2. *A preference relation $\succsim$ satisfies* **B1** *and* **A2–8** *if and only if there exist a non-constant, continuous and affine function $u : X \to \mathbb{R}$ and nonempty, closed and convex sets $C^\sharp, C^\flat \subseteq \Delta(\Omega)$ such that*

$$f \succsim g \Longleftrightarrow \left[\min_{\mu \in C^\flat} \int (u \circ f) d\mu \geq \max_{\mu \in C^\sharp} \int (u \circ g) d\mu \quad or \quad f = g\right].$$

Moreover, $C^\sharp$ and $C^\flat$ are unique, and u is unique up to positive affine transformations.

The next result shows that the disjointness of $C^\sharp$ and $C^\flat$ is equivalent to the transitivity of $\succsim$. In particular, since Lemma 4 deals with a more general class of preferences than Theorem 2, the next proposition inherits the results of Lemma 5 as well.

Proposition 9. *Suppose that $\succsim$ satisfies* **B1** *and* **A2–8***. Let $(u, C^\sharp, C^\flat)$ be an arbitrary profile that represents $\succsim$ in the way of Theorem 2. The followings are equivalent:*

(i) $C^\sharp \cap C^\flat \neq \emptyset$*;*

(ii) $\succsim$ *satisfies transitivity;*

(iii) *For any $f \in \mathcal{F}$,*

$$\max_{\mu \in C^\sharp} \int (u \circ f) d\mu \geq \min_{\mu \in C^\flat} \int (u \circ f) d\mu; \text{ and}$$

(iv) *For any $f, g \in \mathcal{F}$ such that $f \succsim g$,*

$$\left(\max_{\mu \in C^\sharp} \int (u \circ f) d\mu - \max_{\mu \in C^\flat} \int (u \circ g) d\mu \right) \left(\min_{\mu \in C^\sharp} \int (u \circ f) d\mu - \min_{\mu \in C^\flat} \int (u \circ g) d\mu \right) \geq 0.$$

Proof. The statements (ii), (iii), and (iv) are equivalent due to Lemma 5. Also, it is straightforward to see (i) ⇒ (iii). Finally, we can show (iii) ⇒ (i) by appealing to the separating hyperplane theorem. □

D.2 An Example of Not C-continuous Completions

We show that C-continuity in Proposition 5 is an indispensable assumption by presenting an example of completion $\succsim^*$ that cannot be represented as generalized α-maximin.

Let $\succsim$ be represented by $(u, C^\sharp, C^\flat)$. For simplicity, assume that $C \equiv C^\sharp = C^\flat$, $|C| \neq 1$, and $X = [0, 1]$. We also assume that $u : [0, 1] \to \mathbb{R}$ is strictly increasing. Now, consider the following two utility functions on acts:

$$I(f) = \alpha \min_{\mu \in C} \int (u \circ f) d\mu + (1 - \alpha) \max_{\mu \in C} \int (u \circ f) d\mu,$$
$$J(f) = \beta \min_{\mu \in C} \int (u \circ f) d\mu + (1 - \beta) \max_{\mu \in C} \int (u \circ f) d\mu,$$

where $0 \leq \beta < \alpha \leq 1$. By Proposition 5, the complete preferences derived from I and J are completions of $\succsim$.

Beside these completions, we consider a weak order $\succsim^*$ defined as follows:

$$f \succsim^* g \Longleftrightarrow \begin{cases} I(f) > I(g); \text{ or} \\ I(f) = I(g) \text{ and } J(f) \geq J(g). \end{cases}$$

It is not hard to check that $\succsim^*$ is a completion of $\succsim$, while we claim that $\succsim^*$ violates C-continuity. To see this, let f be a non-constant act such that $\min_{\mu \in C} \int (u \circ f) d\mu < \max_{\mu \in C} \int (u \circ f) d\mu$.[31] Then, $x \in \mathcal{L}^c(f)$ holds whenever $u(x) < I(f)$, and $x \in \mathcal{U}^c(f)$ holds whenever $u(x) > I(f)$. What if $u(x) = I(f)$? Since $I(f) < J(f)$ and $u(x) = I(x) = J(x)$, we have $f \succsim^* x$ but $x \not\succsim^* f$. Therefore, $\mathcal{L}^c(f) = \big[0, u^{-1}(I(f))\big]$ and $\mathcal{U}^c(f) = \big(u^{-1}(I(f)), 1\big]$, from which we confirm the violation of C-continuity.

D.3 On the Genericity of Non-transparency

As in Section 6 of the main text, let $\Omega = [0, \bar{b}]$ be an interval of bids in a second-price auction, and Σ be its Borel algebra. Slightly changing the notation from the main text, we denote by $\Delta(\Omega)$ the set of all probability measures on (Ω, Σ).[32] As before, $\Delta(\Omega)$ is endowed with the weak-* topology, which is metrized by the Prokhorov metric d^{P}.

Let $\mathcal{C}$ be the family of nonempty, closed, and convex subsets of $\Delta(\Omega)$, and let $\mathcal{C}^*$ collect all pairs $(C^\sharp, C^\flat) \in \mathcal{C}$ satisfying $C^\sharp \cap C^\flat \neq \emptyset$. We endow $\mathcal{C}$ with the Hausdorff metric defined by

$$d^{\mathrm{H}}(C_1, C_2) = \max\left\{ \max_{\mu_1 \in C_1} \min_{\mu_2 \in C_2} d^{\mathrm{P}}(\mu_1, \mu_2),\ \max_{\mu_2 \in C_2} \min_{\mu_1 \in C_1} d^{\mathrm{P}}(\mu_1, \mu_2) \right\},$$

and endow $\mathcal{C}^*$ with the natural extension of d^{H} given by $\max\{d^{\mathrm{H}}(C_1^\sharp, C_2^\sharp),\ d^{\mathrm{H}}(C_1^\flat, C_2^\flat)\}$ to measure the distance between pairs $(C_1^\sharp, C_1^\flat)$ and $(C_2^\sharp, C_2^\flat)$. Slightly abusing the notation, the metric on $\mathcal{C}^*$ is again denoted by d^{H}.

Let $u : [0, \bar{b}]^2 \to \mathbb{R}$ be the DM's ex-post utility function that satisfies all the conditions presented in Section 6. Given $(C^\sharp, C^\flat)$, recall that truth-telling is said to be transparent for the DM if the maximal expected utility over $C^\sharp$ is equal to the minimal expected utility over $C^\flat$. Using this notion, we define the pairs of "collapsed" belief sets by

$$\mathcal{T} = \left\{ (C^\sharp, C^\flat) \in \mathcal{C}^* : \max_{\mu \in C^\sharp} \int_0^v u(1, \omega) d\mu(\omega) = \min_{\mu \in C^\flat} \int_0^v u(1, \omega) d\mu(\omega) \right\}$$

Now, we claim that transparency is a knife-edge condition that is almost surely violated. Formally, the next proposition shows that the complement of the collapsed belief set pairs constitutes a dense subset of $\mathcal{C}^*$.

Proposition 10. *$\mathcal{C}^* \setminus \mathcal{T}$ is dense in $\mathcal{C}^*$ with respect to d^{H}*

[31] Such an act can be found by applying the separating hyperplane theorem to any $\mu, \mu' \in C$ with $\mu \neq \mu'$.
[32] That is, each element of $\Delta(\Omega)$ is countably additive, not just finitely additive.

Proof. Fix any collapsed pair $(C^\sharp, C^\flat) \in \mathcal{T}$ and $\delta > 0$. We are done if there exists a pair $(D^\sharp, D^\flat) \in \mathcal{C}^* \setminus \mathcal{T}$ whose distance from $(C^\sharp, C^\flat)$ is bounded by δ. To this end, let μ_L be a probability measure that assigns a mass to $0 \in [0, \bar{b}]$, and let μ_H be an arbitrary probability measure supported on $(v, \bar{b}]$. Given an arbitrary $\epsilon > 0$, we set

$$D^\sharp = \Big\{(1-\epsilon)\mu + \epsilon_L \mu_L + \epsilon_H \mu_H : \mu \in C^\sharp \text{ and } \epsilon_L, \epsilon_H \geq 0 \text{ with } \epsilon_L + \epsilon_H = \epsilon\Big\},$$
$$D^\flat = \Big\{(1-\epsilon)\mu + \epsilon_L \mu_L + \epsilon_H \mu_H : \mu \in C^\flat \text{ and } \epsilon_L, \epsilon_H \geq 0 \text{ with } \epsilon_L + \epsilon_H = \epsilon\Big\}.$$

Clearly, $D^\sharp$ and $D^\flat$ are non-disjoint, closed, and convex, so that $(D^\sharp, D^\flat) \in \mathcal{C}^*$. Also, note that $\tilde{D}^\sharp \equiv (1-\epsilon)C^\sharp + \epsilon\mu_L \subseteq D^\sharp$ and $\tilde{D}^\flat \equiv (1-\epsilon)C^\flat + \epsilon\mu_H \subseteq D^\flat$ hold. Hence,

$$\begin{aligned}
\max_{\mu \in D^\sharp} \int_0^v u(1,\omega) d\mu(\omega) &\geq \max_{\mu \in \tilde{D}^\sharp} \int_0^v u(1,\omega) d\mu(\omega) \\
&= (1-\epsilon) \max_{\mu \in C^\sharp} \int_0^v u(1,\omega) d\mu(\omega) + \epsilon u(1,0) \\
&> (1-\epsilon) \min_{\mu \in C^\flat} \int_0^v u(1,\omega) d\mu(\omega) + \epsilon \times 0 \\
&= \min_{\mu \in \tilde{D}^\sharp} \int_0^v u(1,\omega) d\mu(\omega) \geq \min_{\mu \in D^\flat} \int_0^v u(1,\omega) d\mu(\omega),
\end{aligned}$$

from which $(D^\sharp, D^\flat) \in \mathcal{C}^* \setminus \mathcal{T}$. Let us show that $d^{\mathrm{H}}(C^\sharp, D^\sharp) < \delta$ when ϵ is small enough. To this end, fix any $\mu^\sharp \in C^\sharp$, and notice that $\mu_\epsilon = (1-\epsilon)\mu + \epsilon_L \mu_L + \epsilon_H \mu_H$ weakly converges to $\mu^\sharp$ as ϵ tends to 0. Hence, for sufficiently small $\epsilon > 0$, we have

$$\min_{\mu \in D^\sharp} d^{\mathrm{P}}(\mu^\sharp, \mu) \leq d^{\mathrm{P}}(\mu^\sharp, \mu_\epsilon) < \delta.$$

In particular, since $C^\sharp$ is weak-* compact, we can pick $\epsilon_1 > 0$ such that the above inequality uniformly holds for any $\mu^\sharp$. Thus, if we use such an ϵ_1 in the definition of $D^\sharp$, then

$$\max_{\mu^\sharp \in C^\sharp} \min_{\mu \in D^\sharp} d^{\mathrm{P}}(\mu^\sharp, \mu) < \delta.$$

Conversely, consider any belief of the form $\mu = (1-\epsilon)\mu^\sharp + \epsilon_L \mu_L + \epsilon_H \mu_H$, which weakly converges to $\mu^\sharp \in C^\sharp$ as ϵ tends to 0. For sufficiently small $\epsilon > 0$, we have

$$\min_{\rho^\sharp \in C^\sharp} d^{\mathrm{P}}(\rho^\sharp, \mu) \leq d^{\mathrm{P}}(\mu^\sharp, \mu) < \delta.$$

Note that the above inequality does hold no matter how ϵ is split into ϵ_L and ϵ_H, provided that $\epsilon > 0$ is small enough. Thus, each $\mu \in D^\sharp$ is parametrized by $\mu^\sharp$. Again, since $C^\sharp$ is weak-* compact, we can pick $\epsilon_2 > 0$ to be used in the definition of $D^\sharp$, so that

$$\max_{\mu^\sharp \in D^\sharp} \min_{\mu \in C^\sharp} d^{\mathrm{P}}(\mu^\sharp, \mu) < \delta.$$

Hence, by choosing $\epsilon < \min\{\epsilon_1, \epsilon_2\}$, we obtain $d^{\mathrm{H}}(C^\sharp, D^\sharp) < \delta$, as desired. Furthermore, the symmetric argument proves $d^{\mathrm{H}}(C^\flat, D^\flat) < \delta$, thereby $d^{\mathrm{H}}((C^\sharp, C^\flat), (D^\sharp, D^\flat)) < \delta$. □